Counting Matrices Over a Finite Field With All Eigenvalues in the Field

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Abstract

Given a finite field \( F \) and a positive integer \( n \), we give a procedure to count the \( n \times n \) matrices with entries in \( F \) with all eigenvalues in the field. We give an exact value for any field for values of \( n \) up to 4, and prove that for fixed \( n \), as the size of the field increases, the proportion of matrices with all eigenvalues in the field approaches \( 1/n! \).

As a corollary, we show that for large fields, almost all matrices with all eigenvalues in the field have all eigenvalues distinct. The proofs of these results rely on the fact that any matrix with all eigenvalues in \( F \) is similar to a matrix in Jordan canonical form, and so we proceed by enumerating the number of \( n \times n \) Jordan forms, and counting how many matrices are similar to each one. A key step in the calculation is to characterize the matrices that commute with a given Jordan form and count how many of them are invertible.

1 Introduction

Let \( F \) be a field and let \( M_n(F) \) denote the set of \( n \times n \) matrices with entries in \( F \). As an example, consider \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R}) \). The roots of its characteristic polynomial \( \det(A - \lambda I) \) are \( \lambda = \pm i \), which are the eigenvalues of \( A \). Though the entries in \( A \) are real numbers, the eigenvalues are not. This example serves to motivate the following question: If \( Eig_n(F) \) denotes the set of elements of \( M_n(F) \) that have all of their eigenvalues in \( F \), what is the cardinality of \( Eig_n(F) \)? For a field like \( \mathbb{R} \) that is uncountably infinite, this question is trivial, but in this paper we examine the case when \( F \) is a finite field with \( q \) elements. This line of research was initiated by Olsavsky [4], who determined that for any prime \( p \),

\[
|Eig_2(\mathbb{Z}_p)| = \frac{1}{2}p^4 + p^3 - \frac{1}{2}p^2. \tag{1}
\]

In this paper we present a method for determining \( |Eig_n(F)| \) for any \( n \). We use the fact that any matrix \( A \in M_n(F) \) with all eigenvalues in \( F \) is similar to a matrix \( J \) in Jordan canonical form. Thus we can determine \( |Eig_n(F)| \) using the following procedure:

1. Enumerate all \( n \times n \) Jordan forms.
2. Enumerate all matrices in $M_n(\mathbb{F})$ that are similar to each Jordan form.

In Section 2, we review the definitions and notation necessary to work with Jordan forms. Then in Section 3 we explain the procedure to determine $|Eig_n(\mathbb{F})|$, giving a general formula in Equation 3. We illustrate the process for the case $n = 2$, giving a slightly shorter derivation of Equation 1 than was given in [4].

We group matrices in Jordan form by what we call their double partition type, which is defined in Section 3. In Sections 4 and 5 we find formulas for the quantities required to compute $|Eig_n(\mathbb{F})|$ for any $n$. In Section 4 we state a formula for the number of Jordan forms of a given double partition type. Then in Section 5 we prove that the number of matrices similar to any matrix in Jordan form depends only on its double partition type, and give a formula that determines this number for any double partition type. In Section 6, we use these results to give explicit formulas for $|Eig_3(\mathbb{F})|$ and $|Eig_4(\mathbb{F})|$ for any finite field $\mathbb{F}$.

In [4], Olsavsky also noted that the proportion of matrices in $M_2(\mathbb{Z}_p)$ with all eigenvalues in $\mathbb{Z}_p$ approaches 1/2 as $p$ goes to infinity. In Section 7, we generalize this result to prove that the proportion of matrices in $M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ approaches $1/n!$ as $q$ approaches infinity. As a corollary, we prove that for large finite fields, if a matrix has all eigenvalues in the field, then almost surely all of its eigenvalues are distinct.

## 2 Jordan canonical form

Denote the set of invertible matrices in $M_n(\mathbb{F})$ by $GL_n(\mathbb{F})$. We will repeatedly use the fact that any element of $M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ is similar to a matrix in Jordan canonical form. Here we review the necessary definitions. For a more thorough introduction, see [2], Chapter 7.4.

For $1 \leq i \leq n$, let $A_i$ be a square matrix. The **direct sum** of these matrices, denoted $\bigoplus_{i=1}^n A_i$, is a block diagonal matrix such that the matrices $A_i$ lie on the diagonal, and all other entries are zero:

$$\bigoplus_{i=1}^n A_i = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{pmatrix}.$$

A **Jordan block** of size $k \geq 1$ corresponding to some eigenvalue $\lambda \in \mathbb{F}$ is a $k \times k$ matrix with $\lambda$'s along the diagonal, 1's along the superdiagonal, and 0's everywhere else (see Figure 1). Let $A, J \in M_n(\mathbb{F})$. Then $A$ is **similar** to $J$ if there exists a matrix $P \in GL_n(\mathbb{F})$ such that $A = PJP^{-1}$. It is well known (see for example Chapter 7.4, Corollary 4.7 (iii) in [2]) that any matrix $A \in M_n(\mathbb{F})$ with all eigenvalues in $\mathbb{F}$ is similar to a matrix $J$ which is the direct sum of Jordan blocks, and this matrix $J$ is unique up to the ordering of the blocks. The **Jordan canonical form** (or simply **Jordan form**) for $A$ is this direct sum of Jordan blocks. We will use $J_n(\mathbb{F})$ to denote the set of Jordan forms in $M_n(\mathbb{F})$. We note that multiple matrices may correspond to a given Jordan form (e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are representatives of the same form), so when we wish to work with a particular matrix in $J_n(\mathbb{F})$, we will specify an order for the blocks in the direct sum. For example, in Figure 2, the matrix $A \in M_4(\mathbb{Z}_5)$ has the Jordan form that is the direct sum of the Jordan blocks.
obtain the following formula for $|B|$

It is well known (see for example Proposition 1.10.1 in [6]) that if $A \in GL_n(F)$, there exist matrices $P$ and $Q$ in $GL_n(F)$ such that $PA = AP$ and $QA = AQ$. We denote the subgroup of $GL_n(F)$ of matrices $P$ such that $PA = AP$ by $GL_n(A)$. For any matrix $A \in M_n(F)$, let $S(A) \subseteq M_n(F)$ be the set of matrices similar to $A$, and let $C(A)$ denote the subgroup of $GL_n(F)$ of matrices $P$ such that $PA = AP$.

**Lemma 3.1** For any $J \in M_n(F)$, $|S(J)| = \frac{|GL_n(F)|}{|C(J)|}$.

**Proof:** Fix a matrix $J \in M_n(F)$. We wish to find the cardinality of $S(J) = \{AJA^{-1} : A \in GL_n(F)\}$. For any $A, B \in GL_n(F)$, $AJA^{-1} = BJB^{-1}$ if and only if $B^{-1}AJ = JB^{-1}A$, i.e. if $B^{-1}A \in C(J)$. Now, $B^{-1}A \in C(J)$ if and only if $A$ and $B$ are in the same coset of $C(J)$ in $GL_n(F)$, and thus $S(J)$ has the same cardinality as the number of cosets of $C(J)$ in $GL_n(F)$, which is equal to $|GL_n(F)|/|C(J)|$ by Lagrange's Theorem.

It is well known (see for example Proposition 1.10.1 in [6]) that if $F$ has $q$ elements, then $|GL_n(F)| = \prod_{i=0}^{n-1}(q^n - q^i)$. Thus to find $|S(J)|$ for any $J$, it suffices to find $|C(J)|$, and we obtain the following formula for $|Eig_n(F)|$.

$$|Eig_n(F)| = \sum_{J \in J_n(F)} |S(J)| = \sum_{J \in J_n(F)} \frac{\prod_{i=0}^{n-1}(q^n - q^i)}{|C(J)|}.$$ (2)
For \( J \in J_n(\mathbb{F}) \), it turns out that \( |C(J)| \) depends on what we will call the double partition type of \( J \), which we define now.

A partition of a positive integer \( n \) is a set of (not necessarily distinct) positive integers \( \{n_1, n_2, \ldots, n_k\} \) such that \( n_1 + n_2 + \cdots + n_k = n \). We define the partition type of a matrix \( J \in J_n(\mathbb{F}) \) as the partition of \( n \) given by the size of the Jordan blocks in \( J \). For example, the partition type of the matrix \( J \in J_4(\mathbb{Z}_5) \) below is \( \{1, 1, 2\} \) since it has two \( 1 \times 1 \) Jordan blocks and one \( 2 \times 2 \) Jordan block.

\[
J = \begin{pmatrix}
3 & 1 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

We define a double partition (or partition of a partition) of a positive integer \( n \) to be a partition where the numbers in the partition are themselves partitioned into subsets. Denote the set of double partitions of \( n \) by \( DP(n) \). For example, \( \{\{2, 3\}, \{1, 1, 3\}, \{2, 3\}, \{4\}\} \in DP(19) \). We define the double partition type (or simply type) of a matrix \( J \in J_n(\mathbb{F}) \) by grouping all elements of its partition type into sets where two elements of the partition are placed in the same set if their corresponding eigenvalues are the same. For example, the double partition type of the matrix \( J \) above is \( \{\{1, 1\}, \{2\}\} \) since the two \( 1 \times 1 \) blocks have the same eigenvalue, and the \( 2 \times 2 \) block has a different eigenvalue. The study of double partitions dates back at least as far as Cayley [1] and Sylvester [7], and the values of \( |DP(n)| \) are collected in the Online Encyclopedia of Integer Sequences sequence A001970 [5].

The utility of knowing the double partition type of a matrix in Jordan form is given by the following lemma.

**Lemma 3.2** If \( J_1, J_2 \in J_n(\mathbb{F}) \) have the same double partition type, then \( |C(J_1)| = |C(J_2)| \).

The proof of Lemma 3.2 is one of the main results of the paper and will be deferred until Section 5. This lemma justifies the following definition: For any double partition type \( T \) define \( c(T) \) and \( s(T) \) so that \( c(T) = |C(J)| \) and \( s(T) = |S(J)| \), where \( J \) is any matrix of type \( T \). Letting \( t(T) \) denote the number of Jordan forms of type \( T \), we can now rewrite Equation 2 as follows.

\[
|Eig_n(\mathbb{F})| = \sum_{J \in J_n(\mathbb{F})} |S(J)|
= \sum_{T \in DP(n)} t(T)s(T) = \sum_{T \in DP(n)} t(T) \frac{\prod_{i=0}^{n-1}(q^n - q^i)}{c(T)}.
\]

We now are prepared to illustrate our procedure for determining \( |Eig_n(\mathbb{F})| \) by computing \( |Eig_2(\mathbb{F})| \). The first step is to enumerate all the double partition types in \( DP(n) \). In the case \( n = 2 \), there are two possible partitions, \( 2 = 1 + 1 \), and \( 2 = 2 \). There are three double partition types, which we denote by \( T_{2,i} \), for \( 1 \leq i \leq 3 \): \( T_{2,1} = \{\{1\}, \{1\}\} \), \( T_{2,2} = \{\{1, 1\}\} \), and \( T_{2,3} = \{\{2\}\} \). We give a general formula for \( t(T) \) in Lemma 4.1, but for \( n = 2 \) we can just examine each type. The Jordan forms of type \( T_{2,1} \) are represented by matrices of the form \( \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) where \( \lambda_1 \neq \lambda_2 \) and thus \( t(T_{2,1}) = \binom{2}{2} \). Jordan forms of type \( T_{2,2} \) are
represented by matrices of the form \((\lambda \ 0\ 0\ \lambda)\), and Jordan forms of type \(T_{2,3}\) are represented by matrices of the form \((\lambda\ 1\ 0\ \lambda)\), so \(t(T_{2,2}) = t(T_{2,3}) = q\).

It remains to compute \(c(T)\) for each double partition type. A general formula for \(c(T)\) is given by Equation 4, but again for \(n = 2\) we can compute it in an ad-hoc manner. A \(2 \times 2\) matrix commutes with a matrix \(J\) of type \(T_{2,1}\) if and only if it is of the form \((a\ 0\ 0\ b)\), where \(a, b \in \mathbb{F}\), and of course this is true regardless of the specific values of \(\lambda_1\) and \(\lambda_2\). Since an element of \(C(J)\) must be invertible, there are \(q - 1\) choices each for \(a\) and \(b\), and \(c(T_{2,1}) = (q - 1)^2\). Similarly, if \(J\) is of type \(T_{2,2}\), \(C(J) = GL_2(\mathbb{F})\), and \(c(T_{2,2}) = (q^2 - 1)(q^2 - q)\). Finally, if \(J\) is of type \(T_{2,3}\), \(C(J) = \{(a\ b) : a, b \in \mathbb{F}, a \neq 0\}\), and so \(c(T_{2,3}) = q(q - 1)\).

To complete the example, we apply Equation 3:

\[
|Eig_2(\mathbb{F})| = \sum_{T \in DP(2)} t(T) \frac{|GL_2(\mathbb{F})|}{c(T)} = 3 \sum_{i=1}^3 t(T_{2,i}) \frac{(q^2 - 1)(q^2 - q)}{c(T_{2,i})} = \frac{q}{2} \left(\frac{(q^2 - 1)(q^2 - q)}{(q - 1)^2}\right) + q \left(\frac{(q^2 - 1)(q^2 - q)}{q(q - 1)}\right) + \frac{q}{q(q - 1)} = \frac{1}{2}q^4 + q^3 - \frac{1}{2}q^2.
\]

4 The number of Jordan forms of a given type

Fix \(n\) and consider a double partition \(T = \{S_1, S_2, \ldots, S_k\}\) of \(n\) where for \(1 \leq i \leq k\), \(S_i\) is a set of positive integers. To determine the number of Jordan forms in \(J_n(\mathbb{F})\) of type \(T\), we count the number of ways to assign eigenvalues to the \(S_i\)'s. Of course, if \(S_i = S_j\) then assigning eigenvalue \(\lambda_1\) to \(S_i\) and \(\lambda_2\) to \(S_j\) yields the same Jordan form as assigning \(\lambda_2\) to \(S_i\) and \(\lambda_1\) to \(S_j\), so we need to account for any repeated subsets in \(T\). For example, if \(n = 19\) and \(T = \{\{2, 3\}, \{1, 1, 3\}, \{2, 3\}, \{4\}\}\), then the subset \(\{2, 3\}\) is repeated twice, and there will be \(\binom{q}{2}(q - 2)(q - 3) = \binom{q}{2}\binom{q-2}{1}\binom{q-3}{1} = \frac{q!}{2!1!1!}!\) elements of \(J_n(\mathbb{F})\) of type \(T\). The value of \(t(T)\) in the general case is given by the following lemma.

**Lemma 4.1** Let \(T = \{S_1, S_2, \ldots, S_k\}\) be a double partition, where for \(1 \leq i \leq k\), \(S_i\) is a set of positive integers. Let \(B_1, \ldots, B_l\) be equivalence classes of the sets in \(T\) so that \(S_i\) and \(S_j\) are in the same equivalence class if and only if \(S_i = S_j\). Let \(b_i = |B_i|\). Then the number of Jordan forms of type \(T\) in \(J_n(\mathbb{F})\) is \(0\) if \(q < k\) and otherwise is given by the formula

\[
t(T) = \binom{q}{b_1}\binom{q-b_1}{b_2}\cdots\binom{q-b_1-b_2-\cdots-b_{l-1}}{b_l} = \frac{q!}{b_1!b_2!\cdots b_l!(q-b_1-b_2-\cdots-b_l)!}.
\]

**Proof:** There are \(\binom{q}{b_1}\) ways to assign eigenvalues to the sets in \(B_1\), \(\binom{q-b_1}{b_2}\) ways to assign eigenvalues to the sets in \(B_2\) without repeating any of the \(b_1\) eigenvalues already assigned to sets in \(B_1\), and so forth. ■
forms, with the ultimate statement of this characterization coming in Corollary 5.3. This
In this section we first characterize the structure of the matrices that commute with Jordan
5 Invertible matrices that commute with a Jordan form
For any matrix $A$
and this is done in Section 5.1.

![Figure 4: Examples of streaky upper triangular matrices](image)

$$J_A = \begin{pmatrix} 
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda 
\end{pmatrix}$$

$$= \begin{pmatrix} 
\lambda a_{1,1} + a_{1,1} & \lambda a_{1,2} + a_{1,2} & \lambda a_{1,3} + a_{1,3} & \lambda a_{1,4} + a_{1,4} & \lambda a_{1,5} + a_{1,5} \\
\lambda a_{2,1} + a_{2,1} & \lambda a_{2,2} + a_{2,2} & \lambda a_{2,3} + a_{2,3} & \lambda a_{2,4} + a_{2,4} & \lambda a_{2,5} + a_{2,5} \\
\lambda a_{3,1} + a_{3,1} & \lambda a_{3,2} + a_{3,2} & \lambda a_{3,3} + a_{3,3} & \lambda a_{3,4} + a_{3,4} & \lambda a_{3,5} + a_{3,5} \\
\lambda a_{4,1} + a_{4,1} & \lambda a_{4,2} + a_{4,2} & \lambda a_{4,3} + a_{4,3} & \lambda a_{4,4} + a_{4,4} & \lambda a_{4,5} + a_{4,5} 
\end{pmatrix}$$

$$J_A$$

$$= \begin{pmatrix} 
\lambda a_{1,1} & \lambda a_{1,2} + a_{1,1} & \lambda a_{1,3} + a_{1,2} & \lambda a_{1,4} + a_{1,3} & \lambda a_{1,5} + a_{1,4} \\
\lambda a_{2,1} & \lambda a_{2,2} + a_{2,1} & \lambda a_{2,3} + a_{2,2} & \lambda a_{2,4} + a_{2,3} & \lambda a_{2,5} + a_{2,4} \\
\lambda a_{3,1} & \lambda a_{3,2} + a_{3,1} & \lambda a_{3,3} + a_{3,2} & \lambda a_{3,4} + a_{3,3} & \lambda a_{3,5} + a_{3,4} \\
\lambda a_{4,1} & \lambda a_{4,2} + a_{4,1} & \lambda a_{4,3} + a_{4,2} & \lambda a_{4,4} + a_{4,3} & \lambda a_{4,5} + a_{4,3} 
\end{pmatrix}$$

$$= \begin{pmatrix} 
\lambda & 1 & 0 & 0 \\
0 & \lambda & 1 & 0 \\
0 & 0 & \lambda & 1 \\
0 & 0 & 0 & \lambda 
\end{pmatrix}$$

$A_J$

$$= \begin{pmatrix} 
\lambda a_{1,1} & \lambda a_{1,2} + a_{1,1} & \lambda a_{1,3} + a_{1,2} & \lambda a_{1,4} + a_{1,3} & \lambda a_{1,5} + a_{1,4} \\
\lambda a_{2,1} & \lambda a_{2,2} + a_{2,1} & \lambda a_{2,3} + a_{2,2} & \lambda a_{2,4} + a_{2,3} & \lambda a_{2,5} + a_{2,4} \\
\lambda a_{3,1} & \lambda a_{3,2} + a_{3,1} & \lambda a_{3,3} + a_{3,2} & \lambda a_{3,4} + a_{3,3} & \lambda a_{3,5} + a_{3,4} \\
\lambda a_{4,1} & \lambda a_{4,2} + a_{4,1} & \lambda a_{4,3} + a_{4,2} & \lambda a_{4,4} + a_{4,3} & \lambda a_{4,5} + a_{4,3} 
\end{pmatrix}$$

Figure 3: Some examples of streaky upper triangular matrices

Figure 4: Examples of $J_A$ and $A_J$ where $A$ is $4 \times 5$ and $J_4$ and $J_5$ are Jordan blocks.

5 Invertible matrices that commute with a Jordan form

In this section we first characterize the structure of the matrices that commute with Jordan
types, and the ultimate statement of this characterization coming in Corollary 5.3. This
characterization depends only on the double partition type, and not the specific eigenvalues,
and this observation is sufficient to prove Lemma 3.2. To determine $|C(J)|$, we must
determine how many of the matrices of the form specified by Corollary 5.3 are invertible,
and this is done in Section 5.1.

For any matrix $A \in M_n(\mathbb{F})$, let $(A)_{i,j}$ denote the entry of $A$ in the $i$th row and $j$th column.
We say an $n \times m$ matrix $A$ is **streaky upper triangular** if it has the following three properties (see Figure 3):

(i) $(A)_{i,1} = 0$ if $i > 1$.

(ii) $(A)_{n,j} = 0$ if $j < m$.

(iii) $(A)_{i,j} = (A)_{i+1,j+1}$ if $1 \leq i \leq n - 1$, $1 \leq j \leq m - 1$. 
We denote the set of \( n \times m \) streaky upper triangular matrices over \( \mathbb{F} \) by \( SUT_{n,m}(\mathbb{F}) \). In Lemmas 5.1 and 5.2 we examine products of the form \( J_nA \) and \( AJ_m \) where \( A \) is \( n \times m \), and \( J_n \) and \( J_m \) are \( n \times n \) and \( m \times m \) Jordan blocks, respectively (see Figure 4). We show \( J_nA = AJ_m \) if and only if \( J_n \) and \( J_m \) have the same eigenvalue and \( A \in SUT_{n,m}(\mathbb{F}) \), or if \( J_n \) and \( J_m \) have different eigenvalues and \( A \) is the all zeros matrix. These lemmas enable us to characterize the matrices that commute with any matrix \( J \in J_n(\mathbb{F}) \).

**Lemma 5.1** Let \( J_n \in J_n(\mathbb{F}) \) and \( J_m \in J_m(\mathbb{F}) \) be Jordan blocks with eigenvalue \( \lambda \), and suppose \( A \) is an \( n \times m \) matrix. Then \( J_nA = AJ_m \) if and only if \( A \in SUT_{n,m}(\mathbb{F}) \).

**Proof:** Suppose \( J_nA = AJ_m \). We will show \( A \in SUT_{n,m}(\mathbb{F}) \).

First, we examine the individual entries in the first column. For \( 1 \leq i \leq n - 1 \), \((J_nA)_{i,1} = \lambda(A)_{i,1} + (A)_{i+1,1} \), and \((AJ_m)_{i,1} = \lambda(A)_{i,1} \). If these two quantities are equal, then \((A)_{i+1,1} = 0 \) for \( 1 \leq i \leq n - 1 \), which implies \( A \) has Property (i).

Next, we examine the individual entries in the last \((n)\)th row. For \( 2 \leq j \leq m \), \((J_nA)_{n,j} = \lambda(A)_{n,j} \), and \((AJ_m)_{n,j} = \lambda(A)_{n,j} + (A)_{n,j-1} \). If these two quantities are equal, then \((A)_{n,j-1} = 0 \) for \( 2 \leq j \leq m \), which implies \( A \) has Property (ii).

Finally, we examine the other entries. If \( 1 \leq i \leq n - 1 \), and \( 2 \leq j \leq m \), \((J_nA)_{i,j} = \lambda(A)_{i,j} + (A)_{i+1,j} \), and \((AJ_m)_{i,j} = \lambda(A)_{i,j} + (A)_{i,j-1} \). If these two quantities are equal, then \((A)_{i,j-1} = (A)_{i,j} \) for \( 1 \leq i \leq n - 1 \), \( 2 \leq j \leq m \), which implies \( A \) has Property (iii).

Conversely, suppose \( A \in SUT_{n,m}(\mathbb{F}) \). Then we verify \( J_nA = AJ_m \) using four cases.

- For \( i = n \), \( j = 1 \), \((J_nA)_{n,1} = \lambda(A)_{n,1} = (AJ_m)_{n,1} \).
- For \( i \leq n - 1 \), \( j = 1 \), \((J_nA)_{i,1} = \lambda(A)_{i,1} + (A)_{i+1,1} = \lambda(A)_{i,1} = (AJ_m)_{i,1} \).
- For \( i = n \), \( j \geq 2 \), \((J_nA)_{n,j} = \lambda(A)_{n,j} = \lambda(A)_{n,j} + (A)_{n,j-1} = (AJ_m)_{n,j} \).
- For \( i \leq n - 1 \), \( 2 \leq j \), \((J_nA)_{i,j} = \lambda(A)_{i,j} + (A)_{i+1,j} = \lambda(A)_{i,j} + (A)_{i,j-1} = (AJ_m)_{i,j} \).

**Lemma 5.2** Let \( J_n \in J_n(\mathbb{F}) \) be a Jordan block with eigenvalue \( \lambda \) and \( J_m \in J_m(\mathbb{F}) \) be a Jordan block with eigenvalue \( \mu \), where \( \lambda \neq \mu \), and suppose \( A \) is an \( n \times m \) matrix. Then \( J_nA = AJ_m \) if and only if \((A)_{i,j} = 0 \) for all \( 1 \leq i \leq n \), \( 1 \leq j \leq m \).

**Proof:** If \((A)_{i,j} = 0 \) for all \( 1 \leq i \leq n \), \( 1 \leq j \leq m \), then \( J_nA = AJ_m \).

Conversely, suppose \( J_nA = AJ_m \). First examine the case \( i = n \), \( j = 1 \): \((J_nA)_{n,1} = \lambda(A)_{n,1} \) and \((AJ_m)_{n,1} = \mu(A)_{n,1} \) implies \((A)_{n,1} = 0 \).

Next we proceed by induction on the first column \((j = 1)\). We know \((A)_{n,1} = 0 \). For any \( k \geq 0 \), assume \((A)_{n-k,1} = 0 \). Then \((J_nA)_{n-(k+1),1} = \lambda(A)_{n-(k+1),1} + (A)_{n-k,1} = \lambda(A)_{n-(k+1),1} \), while \((AJ_m)_{n-(k+1),1} = \mu(A)_{n-(k+1),1} \). If these two quantities are equal, then \((A)_{n-(k+1),1} = 0 \). From this we conclude \((A)_{i,1} = 0 \) for all \( 1 \leq i \leq n \).

Next we apply the same induction argument to the last row \((i = n)\): For any \( j \geq 1 \), assume \((A)_{n,j} = 0 \). Then \((J_nA)_{n,j+1} = \lambda(A)_{n,j+1} = \mu(A)_{n,j+1} = \mu(A)_{n,j+1} \). If these two quantities are equal, then \((A)_{n,j+1} = 0 \). From this we conclude \((A)_{n,j} = 0 \) for all \( 1 \leq j \leq m \).

To complete the proof, we show that for all \( i < n, j > 1 \), if \((A)_{i,j-1} = 0 \) and \((A)_{i+1,j} = 0 \) then \((A)_{i,j} = 0 \). If \( i < n \), \( j > 1 \), then \((J_nA)_{i,j} = \lambda(A)_{i,j} + (A)_{i+1,j} \), while \((AJ_m)_{i,j} = \mu(A)_{i,j} + (A)_{i+1,j} \).
Figure 5 contains an illustration of Corollary 5.3. In this example, \( \mu(A_{i,j}) + (A)_{i,j-1} \). If \( (A)_{i,j-1} = 0 \) and \( (A)_{i+1,j} = 0 \) then \( \lambda(A_{i,j}) = \mu(A_{i,j}) \), which implies \( (A)_{i,j} = 0 \). We have shown if the entries below and to the left of \( (A)_{i,j} \) are both zero, then \( (A)_{i,j} \) is zero as well. Since we know the first column and last row of \( A \) contain only zeros, by induction all other entries must be zero as well.

If \( A \) is a block matrix, let \( A_{i,j} \) denote the block in the \( i \)th row and \( j \)th column.

**Corollary 5.3** Suppose \( n = n_1 + n_2 + \cdots + n_k \), and \( J \in J_n(\mathbb{F}) \) is a Jordan form with partition type \( \{n_1, n_2, \ldots, n_k\} \), i.e. \( J \) is the direct sum of \( k \) Jordan blocks \( J_1 \in J_{n_1}(\mathbb{F}), J_2 \in J_{n_2}(\mathbb{F}), \ldots, J_k \in J_{n_k}(\mathbb{F}) \), with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_k \), some of which may be the same. We can represent \( J \) as a \( k \times k \) block matrix where \( J^{i,i} = J_i \) for \( 1 \leq i \leq k \), and \( J^{i,j} \) is an \( n_i \times n_j \) all-zeros matrix if \( i \neq j \). Then \( JA = AJ \) if and only if \( A \) is a \( k \times k \) block matrix, where for \( 1 \leq i, j \leq k \), the block \( A_{i,j} \) is an element of \( SUT_{n_i, n_j}(\mathbb{F}) \) if \( \lambda_i = \lambda_j \), and otherwise \( A^{i,j} \) is an \( n_i \times n_j \) matrix containing all zeros.

Figure 5 contains an illustration of Corollary 5.3. In this example, \( J \in J_{11}(\mathbb{F}) \), and in the terminology of Corollary 5.3, \( k = 6 \), \( (n_1, n_2, n_3, n_4, n_5, n_6) = (3, 2, 2, 1, 1, 2) \), and \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (1, 1, 1, 4, 4, 0) \).

**Proof of Corollary 5.3:** Decompose \( A \in M_n(\mathbb{F}) \) as a \( k \times k \) block matrix, where block \( A^{i,j} \) is \( n_i \times n_j \). Then the equation \( JA = AJ \) implies \( J_i A^{i,j} = A^{i,j} J_j \). If \( \lambda_i = \lambda_j \), then \( J_i A^{i,j} = A^{i,j} J_j \) if and only if \( A^{i,j} \in SUT_{n_i, n_j}(\mathbb{F}) \) by Lemma 5.1. If \( \lambda_i \neq \lambda_j \), then \( J_i A^{i,j} = A^{i,j} J_j \) if and only if \( A^{i,j} \) contains all zeros, by Lemma 5.2.

We now show that Lemma 3.2 follows from Corollary 5.3.

**Proof of Lemma 3.2:** Implicit in the statement of Corollary 5.3 is the fact that the characterization of the matrices that commute with \( J \in J_n(\mathbb{F}) \) does not depend on the specific eigenvalues of the Jordan blocks, only on which blocks share the same eigenvalue. In other words, it depends only on the double partition type of \( J \). The quantity \( |C(J)| \) equals the number of invertible matrices that fit this characterization, and since the characterization depends only on the double partition type of \( J \), if two Jordan forms \( J_1 \) and \( J_2 \) have the same double partition type, then \( |C(J_1)| = |C(J_2)| \).
and this case, in Figure 6, a block rearranged matrix, we need the following notation: If of times, and thus will ultimately be unchanged. To describe the diagonal blocks of the rows are permuted the same way, the sign of the determinant will change an even number of the determinant. Thus if the columns of a matrix are permuted in some way, and the fact that switching the position of two rows or columns of a matrix changes the sign of a matrix is invertible.

As in the example in Figure 5, the matrices on the diagonal are themselves block matrices where each block is streaky upper triangular, and Lemma 5.4 describes when such a matrix is invertible.

The proof of Lemma 5.4 proceeds by taking any block matrix where each block is streaky upper triangular and permuting the rows and columns to obtain an upper triangular block matrix, which is invertible if and only if the matrices on the diagonal are invertible. We use the fact that switching the position of two rows or columns of a matrix changes the sign of the determinant. Thus if the columns of a matrix are permuted in some way, and the rows are permuted the same way, the sign of the determinant will change an even number of times, and thus will ultimately be unchanged. To describe the diagonal blocks of the rearranged matrix, we need the following notation: If A is a \(b \times b\) block matrix, where each block \(A^{i,j}\) is an element of \(SUT_{n,n}(F)\), then we denote by \(A'\) the \(b \times b\) matrix made up of the entries that are on the diagonal of each block \(A^{i,j}\), i.e. \((A')_{i,j} = (A^{i,j})_{1,1}\). For example, in Figure 6, A is a \(2 \times 2\) block matrix with each block in \(SUT_{3,3}(F)\) (i.e. \(b = 2, n = 3\)) and D is a \(4 \times 4\) block matrix D with each block in \(SUT_{2,2}(F)\) (i.e. \(b = 4, n = 2\)). In this case, \(A' = \left(\begin{smallmatrix} 3 & 0 \\ 1 & 4 \end{smallmatrix}\right)\), and \(D'\) is the \(4 \times 4\) matrix shown in the figure. We use the notation \(\{b_1n_1, \ldots, b_kn_k\}\) to denote a set with \(b_i\) elements equal to \(n_i\) for \(1 \leq i \leq k\).

Lemma 5.4 Let \(\{b_1n_1, \ldots, b_kn_k\}\) be a partition of \(n\), with \(n_1 > n_2 > \cdots > n_k\). Suppose \(A \in M_n(F)\) is a \(k \times k\) block matrix, where \(A^{i,j}\) is \(b_in_i \times b_jn_j\), and each \(A^{i,j}\) can itself be represented as a \(b_i \times b_j\) block matrix, where each block is an element of \(SUT_{n_i,n_j}(F)\). Then A is invertible if and only if \((A^{i,j})'\) is invertible for each \(i, 1 \leq i \leq k\). In fact, \(\det(A) = \prod_{i=1}^k (\det((A^{i,i})'))^{n_i}\).

**Proof:** We go by induction on \(n_1\). If \(n_1 = 1\), then \(A = A^{1,1} = (A^{1,1})'\) and the statement follows. Now suppose \(\{b_1n_1, \ldots, b_kn_k\}\) is a partition of \(n\), with \(n_1 > n_2 > \cdots > n_k\). Let

\[
A = \begin{pmatrix}
3 & 2 & 4 & 0 & 3 & 0 \\
0 & 3 & 2 & 0 & 0 & 3 \\
0 & 0 & 3 & 0 & 0 & 0 \\
1 & 0 & 2 & 4 & 1 & 4 \\
0 & 1 & 0 & 0 & 4 & 1 \\
0 & 0 & 1 & 0 & 0 & 4
\end{pmatrix},
D = \begin{pmatrix}
1 & 1 & 0 & 2 & 3 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\
2 & 1 & 4 & 3 & 3 & 4 & 3 & 4 \\
0 & 2 & 0 & 4 & 0 & 3 & 0 & 3 \\
1 & 1 & 1 & 4 & 2 & 5 & 3 & 1 \\
0 & 1 & 0 & 1 & 0 & 2 & 0 & 3 \\
0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 2
\end{pmatrix},
D' = \begin{pmatrix}
1 & 0 & 3 & 0 \\
2 & 4 & 3 & 3 \\
1 & 1 & 2 & 3 \\
1 & 0 & 1 & 2
\end{pmatrix}
\]

Figure 6: Block matrices A and D where each block is an \(n \times n\) streaky upper triangular matrix.

### 5.1 How to determine \(|C(J)|\)

It remains to explain how to determine \(|C(J)|\) for \(J \in J_n(F)\). Simply characterizing the matrices that commute with \(J\) is not enough, since elements of \(C(J)\) must be invertible. Thus we need to determine which of the matrices of the form given in Corollary 5.3 are invertible. Examining the example in Figure 5, we note that \(A\) can be represented as a block diagonal matrix, where one block is \(7 \times 7\) and the other two blocks are \(2 \times 2\). It turns out that for any \(J \in J_n(F)\), the matrices in \(C(J)\) can be represented as block diagonal matrices, and since the determinant of a block diagonal matrix is the product of the determinants of the matrices on the diagonal, we must characterize when the matrices on the diagonal are invertible. As in the example in Figure 5, the matrices on the diagonal are themselves block matrices where each block is streaky upper triangular, and Lemma 5.4 describes when such a matrix is invertible.
Let $B_0 = B N_0 = 0$, and for $1 \leq j \leq k$, let $B_j$ denote the sum $b_1 + b_2 + \cdots + b_j$, and $B N_j$ denote the sum $b_1 n_1 + b_2 n_2 + \cdots + b_j n_j$. Permute the columns of $A$ so that if $0 \leq i \leq B_k - 1$, and $j$ is the smallest index so that $i < B_j$, column $B N_{j-1} + (i - B_{j-1}) n_j + 1$ becomes column $i + 1$, and all other columns become columns $B_k + 1$ to $n$, in the same order as they were in the original matrix $A$. Permute the rows the same way to obtain a $2 \times 2$ block matrix $D$ with block structure $\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$ (see Figure 7). Since the rows and columns were rearranged symmetrically, $\det(D) = \det(A)$. $D_1$ is a $k \times k$ block upper triangular matrix where for $1 \leq i \leq k$, $(D_1)^{i,i} = (A^{i,i})'$. Since $n_1 > n_2 > \cdots > n_k$ and each block is streaky upper triangular, the $(n - B_k) \times B_k$ matrix $D_3$ contains only zeros. $D_4$ can be represented as a $k \times k$ matrix where $(D_4)^{i,j}$ is $b_i (n_i - 1) \times b_j (n_j - 1)$, and each $(D_4)^{i,j}$ can itself be represented as a $b_i \times b_j$ block matrix, where each block is an element of $S U T_{n_{i-1}, n_{j-1}}(\mathbb{F})$. Furthermore, $((D_4)^{i,j})' = (A^{i,j})'$ for each $i$. Thus by the inductive hypothesis, $\det(D_4) = \prod_{i=1}^k (\det((A^{i,i})'))^{n_i - 1}$. Since the determinant of an upper triangular block matrix is the product of the determinants of the blocks on the diagonal,

$$\det(A) = \det(D_1) \det(D_4) = \prod_{i=1}^k \det((A^{i,i})') \prod_{i=1}^k (\det((A^{i,i})'))^{n_i - 1} = \prod_{i=1}^k (\det((A^{i,i})'))^{n_i}.$$  

The inductive argument in the proof of Lemma 5.4 implies that the columns of $A$ can be permuted to obtain a block upper triangular matrix, where the diagonal blocks consist of $n_i$ copies each of $(A^{i,i})'$, for $1 \leq i \leq k$. As a corollary, this implies that $A^{i,i}$ is invertible if and only if $(A^{i,i})'$ is invertible. For example, in Figure 6, $\det(A) = \det(A')^3$ and $\det(D) = (\det(D'))^2$.

**Corollary 5.5** Let $g(b_1 n_1, \ldots, b_k n_k)$ denote the number of invertible matrices of the type specified in Lemma 5.4. Then

$$g(b_1 n_1, \ldots, b_k n_k) = \left[ \prod_{i=1}^k \left( q^{b_i (n_i - 1)} \prod_{j=0}^{b_i - 1} (q^{b_i} - q^j) \right) \right] \left[ \prod_{1 \leq i < j \leq k} q^{2 b_i b_j n_j} \right].$$

**Proof:** First we fix $i$ and count the number of different possibilities for $A^{i,i}$. Since $(A^{i,i})'$ must be invertible, and $(A^{i,i})'$ is $b_i \times b_i$, there are $|GL_{b_i}(\mathbb{F})| = \prod_{j=0}^{b_i - 1} (q^{b_i} - q^j)$ different ways to choose the elements of $(A^{i,i})'$. For each of the $b_i^2$ blocks in $A^{i,i}$, there are $n_i - 1$ other diagonals whose entries may be nonzero, and they may be chosen arbitrarily, so there are $q^{b_i^2 (n_i - 1)}$ ways to choose these other entries. Thus there are $f(b_i, n_i) = \left( q^{b_i^2 (n_i - 1)} \prod_{j=0}^{b_i - 1} (q^{b_i} - q^j) \right)$ ways to choose the entries in a diagonal block $A^{i,i}$. All of the other entries in $A$ which are not required to be zero may be chosen arbitrarily, subject to the constraint that each block is streaky upper triangular. If $i < j$, an element of $S U T_{n_i, n_j}(\mathbb{F})$ has $n_j$ diagonals that may be nonzero, and for $i \neq j$ there are $b_i b_j$ blocks in each of $S U T_{n_i, n_j}(\mathbb{F})$ and $S U T_{n_j, n_i}(\mathbb{F})$. Thus there are a total of $\prod_{1 \leq i < j \leq k} q^{2 b_i b_j n_j}$ ways to
Figure 7: An illustration of Lemma 5.4 with a $12 \times 12$ matrix $A$. Here $n = 12$ is partitioned as $\{1 \cdot 4, 2 \cdot 3, 1 \cdot 2\}$, i.e. $k = 3$, $(n_1, n_2, n_3) = (4, 3, 2)$, and $(b_1, b_2, b_3) = (1, 2, 1)$. Then $(B_1, B_2, B_3) = (1, 3, 4)$, $(B_{N1}, B_{N2}, B_{N3}) = (4, 10, 12)$, and in the permutation, columns 1, 5, 8, and 11 become columns 1, 2, 3, and 4, with the other columns becoming columns 5 through 12, keeping their original order. The rows are permuted the same way. In this example, $(A^{1,1})' = (1)$, $(A^{2,2})' = (1440)$, and $(A^{3,3})' = (3)$, and $\det(A) = 1^4 \cdot (\det(1440))^3 \cdot 3^2$. 
choose these entries. We conclude
\[
g(b_1 n_1, \ldots, b_k n_k) = \prod_{i=1}^{k} f(b_i, n_i) \left[ \prod_{1 \leq i < j \leq k} q^{2 b_i b_j n_j} \right] = \left[ \prod_{i=1}^{k} \left( q^{\sum_{j=0}^{b_i-1} (q^{n_j} - q^j)} \right) \right] \left[ \prod_{1 \leq i < j \leq k} q^{2 b_i b_j n_j} \right].
\]

Now we determine \(|C(J)|\) for any \(J \in J_n(F)\). Any matrix in \(C(J)\) can be represented as a block diagonal matrix, where each block on the diagonal is of the type specified in Lemma 5.4. To be precise, let \(n = n_1 + n_2 + \cdots + n_l\), and \(J \in J_n(F)\) have double partition type
\[
T = \{b_1 n_1, \ldots, b_k n_k \}, \{b_1 n_2, \ldots, b_k n_2 \}, \ldots, \{b_1 n_l, \ldots, b_k n_l \},
\]
where for \(1 \leq \alpha \leq l\), \(n_{\alpha} = \sum_{i=1}^{k} b_{\alpha,i} n_{\alpha,i}\), and \(n_{\alpha,i} > n_{\alpha,j}\) if \(i < j\). Denote the eigenvalue associated with the set \(\{b_{\alpha,1} n_{\alpha,1}, \ldots, b_{\alpha,k} n_{\alpha,k}\}\) by \(\lambda_{\alpha}\). We can represent \(J\) as an \(l \times l\) block diagonal matrix where \(J^{a,\alpha}\) contains the Jordan blocks with eigenvalue \(\lambda_{\alpha}\). If \(A\) commutes with \(J\), then \(A\) can be represented as an \(l \times l\) block matrix where the block \(A^{a,\beta}\) is \(n_{\alpha} \times n_{\beta}\), and Lemma 5.2 guarantees that all off-diagonal blocks \(A^{a,\beta}, \alpha \neq \beta\) contain all zeros. So \(A\) is a block diagonal matrix, which is invertible if and only if each block on the diagonal is itself an invertible matrix. Thus to determine the number of invertible matrices \(A \in C(J)\), it suffices to determine for each \(1 \leq \alpha \leq l\) the number of invertible matrices \(A^{a,\alpha}\).

The matrix \(A^{a,\alpha}\) can be represented as a \(k_{\alpha} \times k_{\alpha}\) block matrix, where \((A^{a,\alpha})^{i,j}\) is \(b_{\alpha,i} n_{\alpha,i} \times b_{\alpha,j} n_{\alpha,j}\), and is itself a \(b_{\alpha,i} \times b_{\alpha,j}\) block matrix where Lemma 5.1 implies each block is an element of \(SUT_{n_{\alpha,i}, n_{\alpha,j}}(F)\). By Lemma 5.4 it is invertible if and only if the diagonal blocks \((A^{a,\alpha})^{i,i}\) are invertible. Corollary 5.5 implies the number of invertible matrices that have the required block structure is given by

\[
|C(J)| = \prod_{\alpha=1}^{l} g(b_{\alpha,1} n_{\alpha,1}, \ldots, b_{\alpha,k_{\alpha}} n_{\alpha,k_{\alpha}}) = \prod_{\alpha=1}^{l} \left( \prod_{i=1}^{k_{\alpha}} \left( q^{\sum_{j=0}^{b_{\alpha,i}-1} (q^{n_{\alpha,i}} - q^j)} \right) \right) \left( \prod_{1 \leq i < j \leq k_{\alpha}} q^{2 b_{\alpha,i} b_{\alpha,j} n_{\alpha,j}} \right).
\]

Using the matrix \(J\) from Figure 5 as an example, in the terminology of Equation 4, \(l = 3\), \((n_1, n_2, n_3) = (7, 2, 2)\), \((k_1, k_2, k_3) = (2, 1, 1)\) and \(n_1 = 7 = 3 + 2 \cdot 2 = b_{1,1} n_{1,1} + b_{1,2} n_{1,2}\), \(n_2 = 2 \cdot 2 = b_{2,1} n_{2,1}\), and \(n_3 = 2 \cdot 2 = b_{3,1} n_{3,1}\). Thus for the matrix \(A\) in the figure to be invertible, there are \(|GL_1(F)| = q - 1\) choices each for \(a\) and \(j\), \(|GL_2(F)| = (q^2 - 1)(q^2 - q)\) choices each for \(\{d, q, w, f\}\) and \(\{h, y, z, i\}\), and \(q\) choices for each other letter. Thus \(|C(J)| = (q-1)^2((q^2-1)(q^2-q))^2 q^{16}\).

### 6 Eig3(F) and Eig4(F)

In this section we compute \(|Eig_3(F)|\) and \(|Eig_4(F)|\) for any field \(F\) with \(q\) elements.
Using Equation 3, we conclude

For $T_{3,1}$

Theorem 6.1

The number of matrices with entries in $\mathbb{F}$ whose eigenvalues are all in $\mathbb{F}$ is

$$|Eig_3(\mathbb{F})| = \frac{1}{6}q^9 + \frac{5}{6}q^8 + \frac{2}{3}q^7 - \frac{1}{6}q^6 - \frac{5}{6}q^5 + \frac{1}{3}q^4.$$ 

Proof: There are three partitions of 3: $3 = 1 + 1 + 1 = 1 + 2 = 3$, and 6 double partitions: $T_{3,1} = \{(1, 1), (1, 1)\}, T_{3,2} = \{(1, 1), (1, 1)\}, T_{3,3} = \{(1, 1), (1, 1)\}, T_{3,4} = \{(2), (1)\}, T_{3,5} = \{(2, 1)\}, T_{3,6} = \{(3)\}$.

For $1 \leq i \leq 6$, Lemma 4.1 gives the value of $t(T_{3,i})$. Corollary 5.3 gives the form of $C(J)$ for $J \in T_{3,i}$, and the value of $c(T_{3,i})$ is determined by Equation 4. This information is summarized in Table 1.

Using Equation 3, we conclude

$$|Eig_3(\mathbb{F})| = \sum_{i=1}^{6} \frac{t(T_{3,i})(q^3 - 1)(q^3 - q)(q^3 - q^2)}{c(T_{3,i})} = \frac{1}{6}q^9 + \frac{5}{6}q^8 + \frac{2}{3}q^7 - \frac{1}{6}q^6 - \frac{5}{6}q^5 + \frac{1}{3}q^4.$$ 

Theorem 6.2

The number of matrices with entries in $\mathbb{F}$ whose eigenvalues are all in $\mathbb{F}$ is

$$|Eig_4(\mathbb{F})| = \frac{1}{24}q^{16} + \frac{3}{8}q^{15} + \frac{11}{12}q^{14} + \frac{5}{8}q^{13} - \frac{1}{4}q^{12} - \frac{1}{8}q^{11} - \frac{5}{12}q^{10} + \frac{3}{8}q^9 + \frac{5}{24}q^8 - \frac{1}{4}q^7 - \frac{1}{2}q^6.$$
Proof: There are five partitions of 4: \( 4 = 1 + 1 + 1 + 1 = 1 + 1 + 2 = 2 + 2 = 1 + 3 = 4 \), and 14 double partitions: \( T_{4,1} = \{\{1\}\}, \{\{1\}\}, \{\{1\}\}, \{\{1\}\} \), \( T_{4,2} = \{\{1\}, \{1\}, \{1\}\} \), \( T_{4,3} = \{\{1,1\}, \{1\}\} \), \( T_{4,4} = \{\{1,1,1\}, \{1\}\} \), \( T_{4,5} = \{\{1,1,1\}, \{1\}\} \), \( T_{4,6} = \{\{2\}, \{1\}, \{1\}\} \), \( T_{4,7} = \{\{2,1\}, \{1\}\} \), \( T_{4,8} = \{\{2,1\}, \{1\}\} \), \( T_{4,9} = \{\{2,1,1\}\} \), \( T_{4,10} = \{\{2\}, \{2\}\} \), \( T_{4,11} = \{\{2,2\}\} \), \( T_{4,12} = \{\{3\}, \{1\}\} \), \( T_{4,13} = \{\{1,3\}\} \), \( T_{4,14} = \{\{4\}\} \). The proof proceeds in the same way as the proof of Theorem 6.1, with the relevant information summarized in the Appendix in Table 2.

7 The Proportion of Matrices with All Eigenvalues in \( \mathbb{F} \)

Olsavský noted [4] that as the size of the field \( \mathbb{Z}_p \) increases, the proportion of matrices in \( M_2(\mathbb{Z}_p) \) with all eigenvalues in \( \mathbb{Z}_p \) approaches 1/2, i.e.

\[
\lim_{p \to \infty} \frac{|Eig_2(\mathbb{Z}_p)|}{|M_2(\mathbb{Z}_p)|} = \lim_{p \to \infty} \frac{1}{2} p^4 + p^3 - \frac{1}{2} p^2.
\]

We generalize this result to show the proportion of matrices with all eigenvalues in \( \mathbb{F} \) for any fixed \( n \) approaches 1/n! (Note that the leading coefficients for the polynomials \(|Eig_3(\mathbb{F})|\) and \(|Eig_4(\mathbb{F})|\) are 1/3! and 1/4!, respectively, so the generalization is true for these cases).

**Theorem 7.1** Let \( \mathbb{F} \) be a finite field with \( q \) elements. Then

\[
\lim_{q \to \infty} \frac{|Eig_n(\mathbb{F})|}{|M_n(\mathbb{F})|} = \frac{1}{n!}.
\]

Proof: We know \(|M_n(\mathbb{F})| = q^{n^2}\) and our method for determining \(|Eig_n(\mathbb{F})|\) implies that it is a polynomial in the variable \( q \): Denote the double partitions in \( DP(n) \) by \( T_{n,i} \) for \( 1 \leq i \leq |DP(n)| \). For each \( i \), \( t(T_{n,i}) \) is a polynomial, and since for each \( J \in J_n(\mathbb{F}) \), \( C(J) \) is a subgroup of \( GL_n(\mathbb{F}) \), the polynomial \(|GL_n(\mathbb{F})|\) is divisible by \(|C(J)|\), and thus \( s(T_{n,i}) \) is also a polynomial. To evaluate the limit, we must determine the leading coefficient of \(|Eig_n(\mathbb{F})|\). Let \( \deg(f) \) denote the degree (in the variable \( q \)) of the polynomial \( f \). Then, since the degree of a sum of a fixed number of polynomials is equal to the maximum degree of the polynomials,

\[
\deg(|Eig_n(\mathbb{F})|) = \deg \left( \sum_{i=1}^{|DP(n)|} t(T_{n,i})s(T_{n,i}) \right)
\]

\[
= \max_{1 \leq i \leq |DP(n)|} \deg(t(T_{n,i})s(T_{n,i}))
\]

\[
= \max_{1 \leq i \leq |DP(n)|} \deg \left( \frac{t(T_{n,i})|GL_n(\mathbb{F})|}{c(T_{n,i})} \right)
\]

\[
= \max_{1 \leq i \leq |DP(n)|} \deg(t(T_{n,i})) + \deg(|GL_n(\mathbb{F})|) - \deg(c(T_{n,i}))
\]

\[
= \max_{1 \leq i \leq |DP(n)|} n^2 + \deg(t(T_{n,i})) - \deg(c(T_{n,i}))
\]

Let \( T_{n,1} \) be the double partition \( \{\{1\}, \{1\}, \ldots, \{1\}\} \), corresponding to the type of Jordan form with all \( n \) eigenvalues distinct. For all \( n \), we will show the maximum is attained.
only for \( i = 1 \). Indeed, since \( \deg(t(T_{n,i})) \) is equal to the number of distinct eigenvalues in the type, \( \deg(t(T_{n,1})) = n \), and \( \deg(t(T_{n,i})) < n \), for all \( i > 1 \). Furthermore, by Equation 4, \( \deg(c(T_{n,1})) = n \), and \( \deg(c(T_{n,i})) \geq n \) for all \( i > 1 \). Thus \( \deg(|\text{Eig}_n(\mathbb{F})|) = n^2 + \deg(t(T_{n,1})) - \deg(c(T_{n,1})) = n^2 \), and the leading coefficient of \( |\text{Eig}_n(\mathbb{F})| \) is equal to the leading coefficient of \( t(T_{n,1}) \cdot |\text{GL}_n(\mathbb{F})|/c(T_{n,1}) \).

Since \( t(T_{n,1}) = \binom{n}{1} \), \( |\text{GL}_n(\mathbb{F})| = \prod_{i=0}^{n-1}(q^n - q^i) \), and \( c(T_{n,1}) = (q-1)^n \), the leading coefficient of \( t(T_{n,1}) \cdot |\text{GL}_n(\mathbb{F})|/c(T_{n,1}) \) is \( 1/n! \).

As a corollary, we note that for large enough finite fields, nearly all of the matrices with all eigenvalues in the field have all different eigenvalues.

**Corollary 7.2** Let \( \text{Eff}_n(\mathbb{F}) \subseteq \text{Eig}_n(\mathbb{F}) \) denote the set of matrices in \( M_n(\mathbb{F}) \) with all eigenvalues in \( \mathbb{F} \) and all eigenvalues distinct. Then

\[
\lim_{q \to \infty} \frac{|\text{Eff}_n(\mathbb{F})|}{|\text{Eig}_n(\mathbb{F})|} = 1.
\]

**Proof:** In the notation of the proof of Theorem 7.1,

\[
|\text{Eff}_n(\mathbb{F})| = t(T_{n,1}) \cdot |\text{GL}_n(\mathbb{F})|/c(T_{n,1}) = \frac{1}{n!}q^{n^2} + o(q^{n^2}).
\]

Thus \( |\text{Eff}_n(\mathbb{F})| \) and \( |\text{Eig}_n(\mathbb{F})| \) are both polynomials in \( q \) with the same leading term. So as \( q \) increases the ratio of these cardinalities will approach 1.

**8 Conclusions**

Given a finite field \( \mathbb{F} \) and a positive integer \( n \), we have given a method for computing \( |\text{Eig}_n(\mathbb{F})| \) using Equation 3. In Lemma 4.1 and Equation 4 we gave the formulas necessary for computing the pieces of Equation 3. In Section 6 we applied our method in the cases \( n = 3 \) and \( n = 4 \), and in Section 7, we showed that as the size of the finite field increases, the proportion of matrices in \( M_n(\mathbb{F}) \) that have all eigenvalues in \( \mathbb{F} \) approaches \( 1/n! \).

There are a number of interesting directions for future research. For example, Theorem 7.1 describes the asymptotic behavior of \( |\text{Eig}_n(\mathbb{F})| \) in the case where \( n \) is fixed and \( q \) goes to infinity. Is it possible to find an analogous statement in the case that \( q \) is fixed and \( n \) increases? It could also be interesting to find a geometric or combinatorial interpretation for these numbers. In the case \( q = 2 \), and \( n = 1, 2, 3, 4 \), \( |\text{Eig}_n(\mathbb{Z}_2)| = 2, 14, 352, \) and 33,632, respectively, and this sequence (or even a small piece of it) is not found in the Online Encyclopedia of Integer Sequences. Finally the polynomials for \( |\text{Eig}_n(\mathbb{F})| \) could be studied further. For example, is the smallest nonzero power in \( |\text{Eig}_n(\mathbb{F})| \) always \( 2(n - 1) \) and if so, why? It seems likely that the coefficients will always sum to 1, but will they always be between -1 and 1, and if so, why? Are there patterns in in the magnitudes or signs of the coefficients?

**9 Acknowledgment**

Thanks to Pamela Richardson for suggesting this problem and for helpful conversations.
References


10 Appendix

\[
\begin{array}{cccc}
T_{4,1} & T_{4,2} & T_{4,3} & T_{4,4} \\
\left\{ \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \right\} & \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \right\} & \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\} & \left\{ \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\} \\
& \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \right\} & \left\{ \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix} \right\} & \left\{ \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ 0 & 0 & 0 & j \end{bmatrix} \right\} \\
& \left\{ \begin{bmatrix} (q^2 - q)(q^2 - 1)(q - 1)^2 \right\} & \left\{ \begin{bmatrix} (q^2 - q)^2(q^2 - 1)^2 \right\} & \left\{ \begin{bmatrix} (q^3 - 1)(q^3 - q)(q^3 - q^2)(q - 1) \right\} \end{array}
\]
\[
\begin{align*}
T_{4,5} & \quad \left\{ \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\} \quad q \quad GL_4(\mathbb{F}) \quad (q^4 - 1)(q^4 - q)(q^4 - q^2)(q^4 - q^3) \\
T_{4,6} & \quad \left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \right\} \quad (\binom{q}{2}(q - 2) \left\{ \begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \right\} \quad q(q - 1)^3 \\
T_{4,7} & \quad \left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \right\} \quad q(q - 1) \left\{ \begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & e \end{bmatrix} \right\} \quad q(q - 1)(q^2 - 1)(q^2 - q) \\
T_{4,8} & \quad \left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\} \quad q(q - 1) \left\{ \begin{bmatrix} a & b & c & 0 \\ 0 & a & 0 & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix} \right\} \quad q^3(q - 1)^3 \\
T_{4,9} & \quad \left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\} \quad q \left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & 0 & 0 \\ 0 & e & f & g \\ 0 & h & i & j \end{bmatrix} \right\} \quad (q^2 - q)(q^2 - 1)(q - 1)q^5 \\
T_{4,10} & \quad \left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\} \quad (\binom{q}{2}) \left\{ \begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & e \end{bmatrix} \right\} \quad q^2(q - 1)^2 \\
T_{4,11} & \quad \left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\} \quad q \left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & 0 & 0 \\ e & f & g & h \\ 0 & e & 0 & g \end{bmatrix} \right\} \quad (q^2 - q)(q^2 - 1)q^4 \\
T_{4,12} & \quad \left\{ \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \right\} \quad q(q - 1) \left\{ \begin{bmatrix} a & b & c & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{bmatrix} \right\} \quad q^2(q - 1)^2 \\
T_{4,13} & \quad \left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\} \quad q \left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{bmatrix} \right\} \quad q^4(q - 1)^2 \\
T_{4,14} & \quad \left\{ \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\} \quad q \left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} \right\} \quad q^3(q - 1)
\end{align*}
\]
Table 2: Values of $t(T_{4,i})$ and $c(T_{4,i})$