Math 150 Spring 14 Exam 1

Name: _________________________

Score: __________

Partial Credit: Please show all work. No credit will be awarded for an answer that is not supported with appropriate work. You must show all steps to get full credit.

True-False: Circle T if the statement is always true. Otherwise circle F. No supporting work will be graded for these problems.

Fill in the Blank: Write the answer that correctly completes the statement. No supporting work will be graded for these problems.

You may NOT use a calculator, electronic devices, book, or notes.

Please turn off all cell phones during the exam.

Please do not talk during the exam.

NO credit will be given for using l’Hospital’s rule.
1. (7 pts) Sketch the graph of ONE function that satisfies ALL of the following:

a. \( \lim_{x \to 3} f(x) = 4 \)  
b. \( \lim_{x \to 3} f(x) = 1 \)  
c. \( \lim_{x \to -3} f(x) = 1 \)  
d. \( \lim_{x \to -3^-} f(x) = \infty \)  
e. \( \lim_{x \to -1} f(x) = 2 \)  
f. \( f(3) = 2 \)  
g. \( f(-1) = 1 \)

Each small square is one unit by one unit
2. (8 pts) Evaluate each of the following limits.

a. \[ \lim_{x \to 3} \frac{\sqrt{2x - 1} + 4}{x - 1} \]

\[ \lim_{x \to 3} \frac{\sqrt{2(3) - 1} + 4}{3 - 1} = \frac{\sqrt{9} + 4}{2} = \frac{3 + 4}{2} = \frac{7}{2} \]

b. \[ \lim_{x \to 2} \frac{x^3 - 2x - 4}{x - 2} \]

\[ \lim_{x \to 2} \frac{x^3 - 2x - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 2)}{x - 2} = \lim_{x \to 2} \left( x^2 + 2x + 2 \right) = (2)^2 + 2(2) + 2 = 10 \]
3. (8 pts) Evaluate each of the following limits.

a. \[ \lim_{x \to 0} \frac{\sin(2x) \cos(3x)}{\tan(5x)} \]

\[
\lim_{x \to 0} \frac{\sin(2x) \cos(3x)}{\tan(5x)} = \lim_{x \to 0} \frac{\sin(2x) \cos(3x)}{1} \cdot \frac{1}{\cos(5x)} = \lim_{x \to 0} \frac{\sin(2x) \cos(3x)}{2x} \cdot \frac{2x}{\cos(5x)} \cdot \frac{1}{\cos(5x)}
\]

\[
= \lim_{x \to 0} \frac{2x}{\cos(5x)} \cdot \frac{1}{\cos(5x)} = \frac{2r}{5}
\]

b. \[ \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} \]

\[
\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} = \lim_{x \to 3} \frac{(\sqrt{x+1} - 2)(\sqrt{x+1} + 2)}{(x-3)(\sqrt{x+1} + 2)}
\]

\[
= \lim_{x \to 3} \frac{x+1 - 4}{(x-3)(\sqrt{x+1} + 2)} = \lim_{x \to 3} \frac{x-3}{(x-3)(\sqrt{x+1} + 2)}
\]

\[
= \lim_{x \to 3} \frac{1}{\sqrt{x+1} + 2} = \frac{1}{\sqrt{3+1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}
\]
4. (6 pts) State the EQUATION involving LIMITS that is used to determine the constant \( c \) that will make the function

\[ f(x) = \begin{cases} 
5x + c, & x \leq 2 \\
x^2 + 2, & x > 2 
\end{cases} \]

continuous everywhere. Then find the value of \( c \).

\[ \lim_{x \to 2} (5x + c) = \lim_{x \to 2} (x^2 + 2) \]

\[ 5(2) + c = (2)^2 + 2 \]

\[ 10 + c = 4 + 2 \]

\[ 10 + c = 6 \]

\[ c = -4 \]
(12 pts) True / False - Circle T if the statement is always true. Otherwise circle F.

T  

a. \( \frac{d}{dx}(\sec x) = \sec x \tan x \)

F  

b. \( \frac{d^{121}}{dx^{121}}(\sin x) = -\cos x \quad \) (Should be \( \cos x \).)

F  

c. If \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \), then \( f \) is continuous at \( x = 2 \).

For the function to be continuous at \( x = 2 \), \( f(2) \) needs to exist.

F  

d. Rational functions are continuous on the interval \( (-\infty, \infty) \).

Rational functions are continuous where they are defined.
\( f(x) = \frac{1}{x-2} \) is NOT defined when \( x = 2 \).

T  

e. If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

T  

f. To use the Intermediate Value Theorem, the function must be continuous on the given interval.

6. (8 pts) Find \( y' \) for each of the following.

**You do not need to simplify.**

a. \( y = (x^3 + 5x + 2)(5x + 7) \)

\( y' = (3x^2 + 5)(5x + 7) + (x^3 + 5x + 2)(5) \)

b. \( y = \frac{x^5 + 2x + 1}{\sqrt{x - 3}} \)

\( y = \frac{(\sqrt{x - 3})(5x^4 + 2) - (x^5 + 2x + 1)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x - 3})^2} \)
7. (10 pts) Limit Definition

a. State the limit definition of the derivative.

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

b. Use the \textit{limit definition of the derivative} to find the derivative of

\[ f(x) = ax^2 + bx + c \]

where \( a, b, \) and \( c \) are constants.

\[
\begin{align*}
  f'(x) &= \lim_{h \to 0} \frac{a(x + h)^2 + b(x + h) + c - (ax^2 + bx + c)}{h} \\
  &= \lim_{h \to 0} \frac{a(x^2 + 2xh + h^2) + b(x + h) + c - (ax^2 + bx + c)}{h} \\
  &= \lim_{h \to 0} \frac{ax^2 + 2axh + ah^2 + bx + bh + c - ax^2 - bx - c}{h} \\
  &= \lim_{h \to 0} \frac{2axh + ah^2 + bh}{h} \\
  &= \lim_{h \to 0} \frac{h(2ax + ah + b)}{h} \\
  &= \lim_{h \to 0} (2ax + ah + b) \\
  &= 2ax + a(0) + b \\
  &= 2ax + b
\end{align*}
\]
8. (15 pts) Find, classify, and verify the discontinuities of the given function.

\[ f(x) = \begin{cases} 
\frac{x^2 - 1}{x - 1}, & x < 3 \\
\frac{x + 2}{x - 5}, & x > 3
\end{cases} \]

Removable discontinuity at \( x = 1 \)

\[ f(1) \text{ d.n.e. and } \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1} (x+1) = 1 + 1 = 2. \]

Infinite discontinuity at \( x = 5 \)

\[ f(5) \text{ d.n.e. } \lim_{x \to 5^-} \frac{x + 2}{x - 5} = -\infty \quad \lim_{x \to 5^+} \frac{x + 2}{x - 5} = +\infty \]

Jump discontinuity at \( x = 3 \)

\[ f(3) \text{ d.n.e.} \]

\[ \lim_{x \to 3^-} \frac{x^2 - 1}{x - 1} = \frac{(3)^2 - 1}{3 - 1} = \frac{9 - 1}{2} = \frac{8}{2} = 4 \]

\[ \lim_{x \to 3^+} \frac{x + 2}{x - 5} = \frac{3 + 2}{3 - 5} = \frac{5}{-2} = \frac{-5}{2} \]
9. (4 pts) (Fill in the blank) For each part, find the missing number.

NO WORK needs to be shown for this problem.

a. STATE the smallest value of \( n \) such that \( \frac{d^n}{dx^n}(x^{103} - 643x^{57} + 13x^{19} + 5) = 0 \).

\[ n = \underline{104} \]

b. If \( f(x) = x^{5/3} \), STATE the smallest value of \( m \) such that \( f^{(m)}(8) < 0 \).

\[ m = \underline{3} \]

\( f'(x) = \frac{5}{3}x^{2/3} \quad f''(x) = \frac{10}{9}x^{-1/3} \quad f'''(x) = -\frac{10}{27}x^{-4/3} \)

\( f'(8) = \frac{5}{3}(8)^{2/3} > 0 \quad f''(8) = \frac{10}{9}(8)^{-1/3} > 0 \quad f'''(8) = -\frac{10}{27}(8)^{-4/3} < 0 \)

10. (6 pts) Find the equation of the tangent line to the curve \( y = 2x^2(\sqrt{x} + 1) \) at \( x = 1 \).

\[ f'(1) = 2(1)^2(\sqrt{1} + 1) = 2(1+1) = 4 \]

So, the tangent line passes through the point \((1, 4)\)

\[ f'(x) = 4x(\sqrt{x} + 1) + 2x^2\left(\frac{1}{2\sqrt{x}}\right) \]

\[ f'(1) = 4(1)(\sqrt{1} + 1) + 2(1)^2\left(\frac{1}{2\sqrt{1}}\right) \]

\[ f'(1) = 4(2) + 2\left(\frac{1}{2}\right) \]

\[ f'(1) = 9 \]

\[ y - y_1 = m(x - x_1) \]

\[ y - 4 = 9(x - 1) \]

\[ y - 4 = 9x - 9 \]

\[ y = 9x - 5 \]
11. (6 pts) State the three conditions for a function $f$ to be continuous at $x = a$.

1. $f(a)$ exists

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

12. (4 pts) If $f(x) = \tan x$, compute $f'(\frac{\pi}{6})$.

$f'(x) = \sec^2(x)$

$f'(\frac{\pi}{6}) = \left(\sec\left(\frac{\pi}{6}\right)\right)^2$

$= \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3}$
13. (6 pts) Verify that \( \frac{d}{dx} (\cot x) = -\csc^2 x \) by expressing \( \cot x \) in terms of \( \sin x \) & \( \cos x \) and using the quotient rule to differentiate.

\[
\frac{d}{dx} (\cot x) = \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right)
= \frac{\sin x (-\sin x) - \cos x (\cos x)}{(\sin x)^2}
= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}
= \frac{-1}{\sin^2 x}
= -\csc^2 x
\]