Turán type problems and polychromatic colorings on the hypercube

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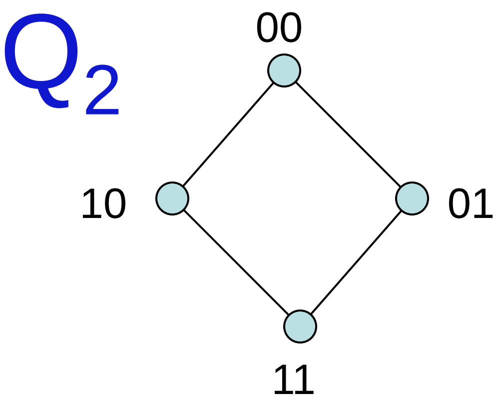
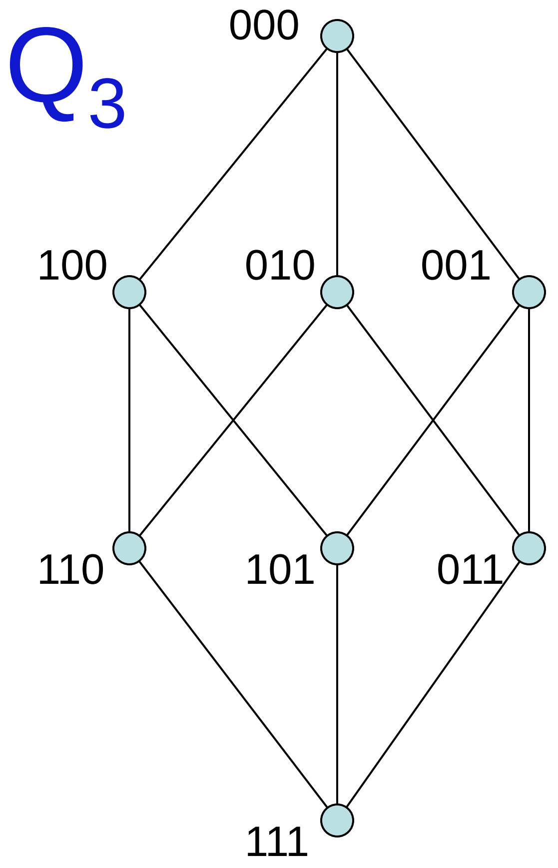
The hypercube Q_n

- $V(Q_n) = \{0,1\}^n$
- $E(Q_n)$ = pairs that differ in exactly one coordinate





Notation for Q_n

- [01★011] denotes the edge between vertices [010011] and [011011]

- [00★★01★] denotes a Q_3 subgraph of Q_7.

Turán type problems

Turán's theorem [1941] answers the question:

- How many edges can an n-vertex graph contain while not containing K_r as a subgraph?

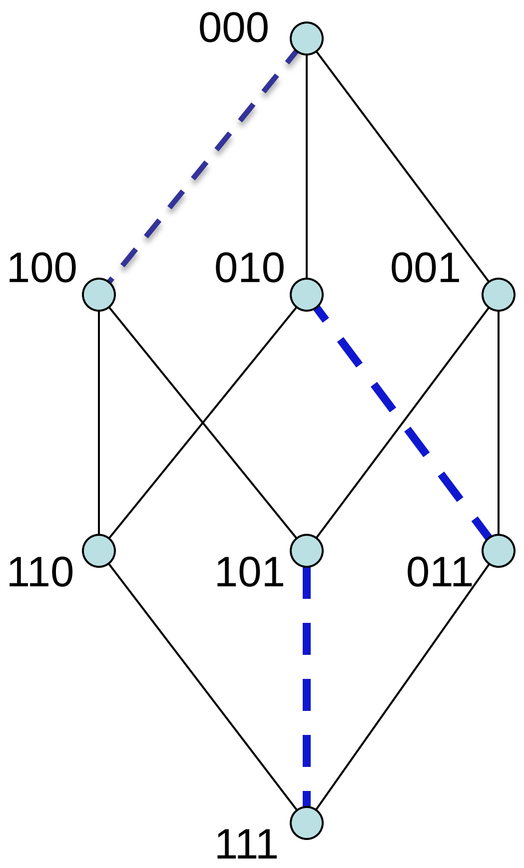
Equivalently:

- What is the minimum number of edges we must delete from K_n to kill all copies of K_r?

Turán type problems on Q_n

c(n,G) = minimum number of edges to delete from Q_n to kill all copies of G.

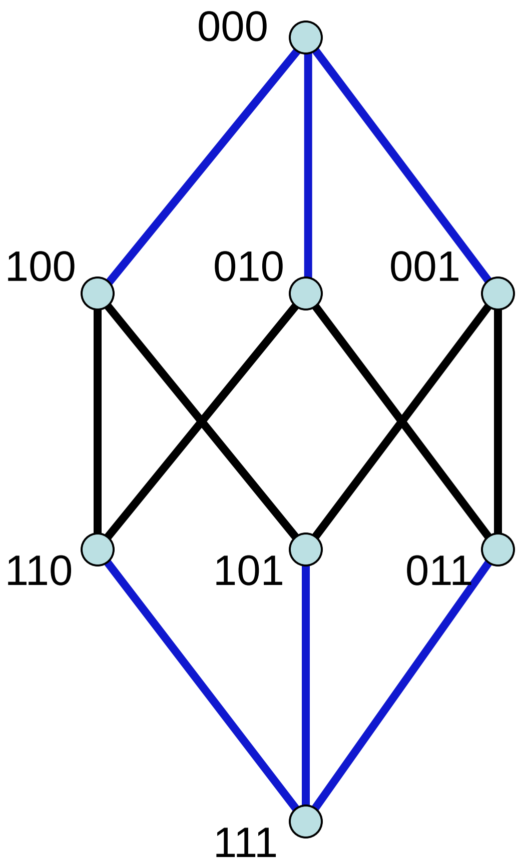
- E.g. $c(3,Q_2) = 3$



$c(G) = \lim_{n\to\infty} c(n,G)/|E(Q_n)|$

Turán type problems on Q_n

Conjecture: $c(Q_2) = 1/2$
[Erdös,1984]



Best known bounds

[Thomason, Wagner, 2008]

[Brass, Harborth, Neinborg, 1995]

$.377 < c(Q_2) < .5(n\text{-}\mathrm{sqrt}(n))2^{n-1}$

Turán type problems on Q_n

- $(\sqrt{2} - 1) \leq c(C_6) \leq 1/3$ [Chung, 1992], [Conder, 1993]
- For $k \geq 2$, $c(C_{4k}) = 1$. [Chung, 1992]
- $c(C_{14}) = 1$. [Füredi, Özkahya, 2009+]
- $c(\text{induced } C_{10}) \leq 1/4$. [Axenovich, Martin, 2006]
- Open: $c(C_{10})$?

Conjecture [Alon, Krech, Szabó, 2007]: $c(Q_3) = 1/4$.

Best known bounds [Offner, 2008]

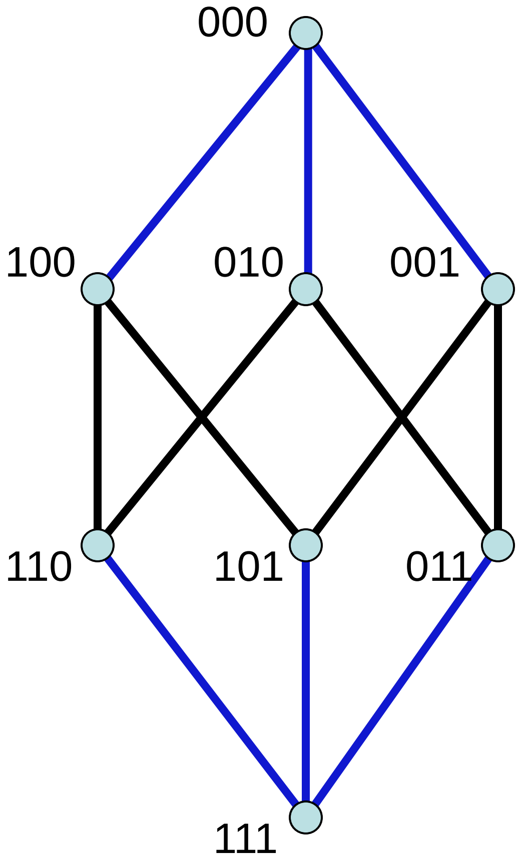
$$.116 < c(Q_3) \leq 1/4$$

Polychromatic colorings

p(G) = maximum number of colors s.t. it is possible to color the edges of any Q_n s.t. every copy of G contains every color.

E.g. $p(Q_2) = 2$.

Motivation: $c(G) \leq 1/p(G)$.



Outline of talk

- Hypercube
- Turan type problems
- Polychromatic colorings

  * Results

  * Lower bounds

  * Upper bounds
- Open problems

Polychromatic colorings

- $d(d+1)/2 \geq p(Q_d) \geq \text{floor}[(d+1)^2/4]$

[Alon, Krech, Szabó, 2007]

- $p(Q_d) = \text{floor}[(d+1)^2/4]$

[Offner, 2008]

We have bounds on p(G) for some other choices of G.

Lower bounds on p(G)

Find a coloring and show every instance of G contains all colors.

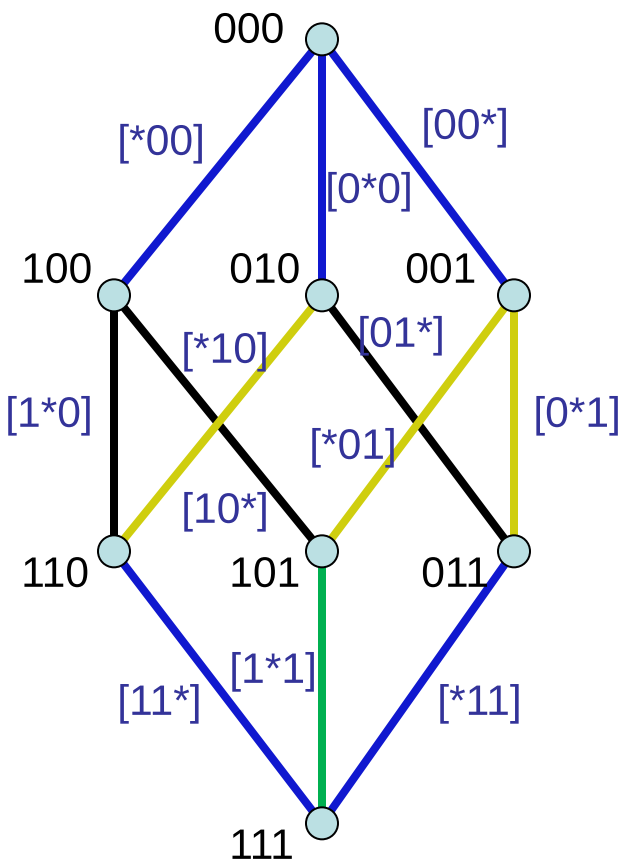
Typical coloring:

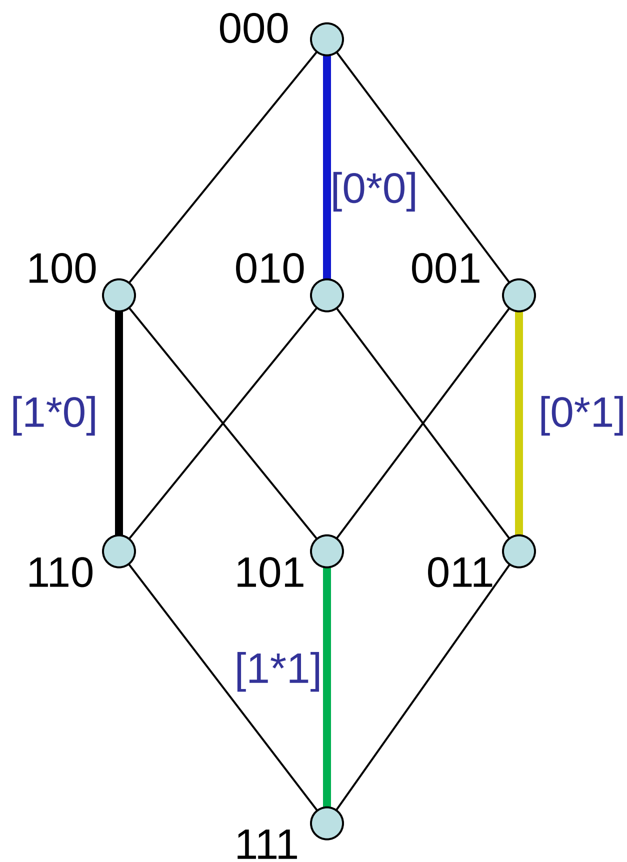
Let l(e) = # of 1s to left of star in edge e.

e.g. l([01★011]) = 1, r([01★011]) = 2.

Lower bounds on p

E.g. $p(Q_3) \geq 4$: Color(e)= (l(e) (mod 2), r(e) (mod 2))





Cube:
[10★★01★]

Edges:
[100★010]
[101★010]
[100★011]
[101★011]

Upper bounds on p: simple colorings

A coloring of Q_n is simple if the color of e is determined by (l(e),r(e)).

Lemma: We need only consider simple colorings.

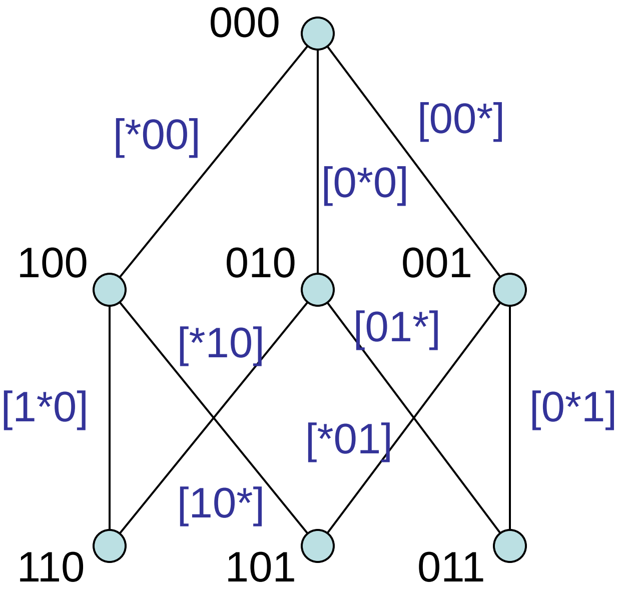
Proof: Application of Ramsey's theorem.

Upper bounds on p

To show $p(G) < r$, show that for any simple r-coloring of Q_n, there is some instance of G containing at most r-1 colors.

e.g. $p(Q_3 \setminus v) < 4$, since there is an instance with edges only in classes

(0,0), (1,0), (0,1).



Upper bounds on p

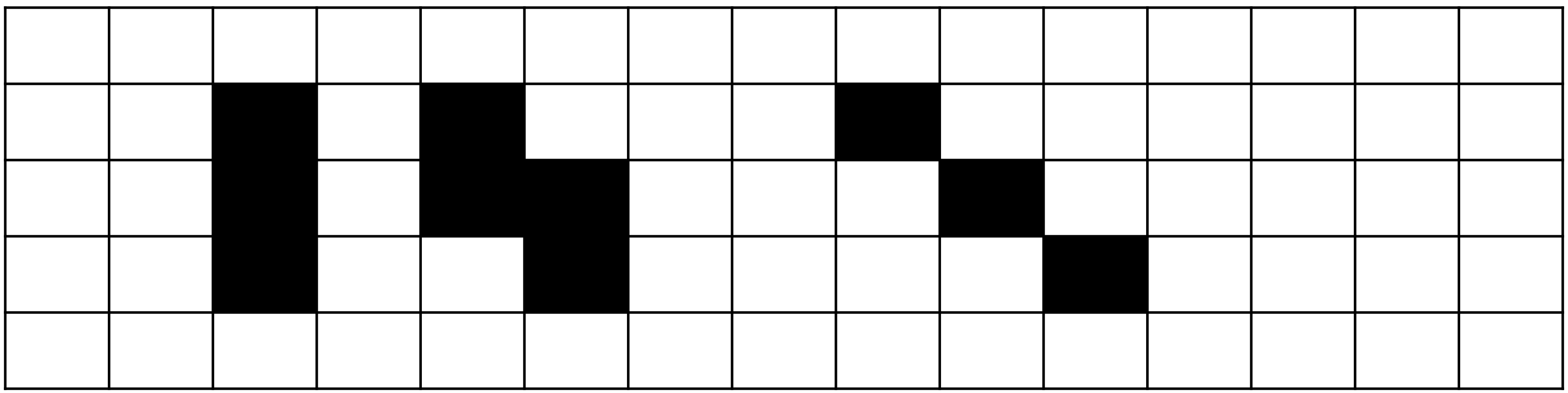
$p(Q_d) \leq \text{floor}[(d+1)^2/4]$.

Idea: Arrange the color classes of Q_n in a grid:

0,0			
0,1	1,0		
0,2	1,1	2,0	
0,3	1,2	2,1	3,0

Examine which color classes are included in copies of Q_d.

Upper bounds on p



Edges using a given star in a Q_d cover a specific "shape" of color classes.

Lemma: For a sequence of shapes where the widest shape in row i has width z_i, the maximum number of colors which can polychromatically color the sequence is at most $\sum z_i$.

Open problems

For which graphs G can we determine p(G)? Right now, Q_d, $Q_d \backslash v$, a few others.

Is it true that for all r, there is some G s.t. p(G) = r?

Open problems

Bialostocki [1983] proved if Q_n is Q_2-polychromatically colored with $p(Q_2)=2$ colors, the color classes are (asymptotically) the same size. Is this true for Q_d, $d>2$?

What can be said about the relationship between p and c?

Thank you.