MTH131 Applied Calculus – Spring 2016
Lab 8 – SOLUTIONS

1. (a) Let $u = x^3 + 4$, $du = 3x^2\, dx$. Subbing in we obtain

$$\begin{aligned}
\int 3x^2\sqrt{x^3+4}\, dx &= \int \sqrt{u}\, du, \\
&= \frac{2}{3}u^{3/2} + C, \\
&= \frac{2}{3}(x^3+4)^{3/2} + C.
\end{aligned}$$

(b) Let $u = 5 - x$, $du = -dx$. Subbing in we obtain

$$\begin{aligned}
\int \frac{1}{5-x}\, dx &= -\int \frac{1}{5-x}(-dx), \\
&= -\int \frac{1}{u}\, du, \\
&= -\ln|u| + C, \\
&= -\ln|5-x| + C.
\end{aligned}$$

(c) Let $u = x^3 - 1$, $du = 3x^2\, dx$. Subbing in we obtain

$$\begin{aligned}
\int x^2(x^3-1)^4\, dx &= \frac{1}{3}\int 3x^2(x^3-1)^4\, dx \\
&= \frac{1}{3}\int u^4\, du \\
&= \frac{1}{3}\left(\frac{1}{5}u^5 + C\right) \\
&= \frac{u^5}{15} + C \\
&= \frac{(x^3-1)^5}{15} + C
\end{aligned}$$

(d) Let $u = x^3 - 3x$, $du = (3x^2 - 3)\, dx = 3(x^2 - 1)\, dx$. Subbing in we obtain

$$\begin{aligned}
\int (x^2-1)(x^3-3x)^4\, dx &= \frac{1}{3}\int 3(x^2-1)(x^3-3x)^4\, dx \\
&= \frac{1}{3}\int u^4\, du \\
&= \frac{1}{3}\left(\frac{1}{5}u^5 + C\right) \\
&= \frac{u^5}{15} + C \\
&= \frac{(x^3-3x)^5}{15} + C
\end{aligned}$$

(e) Let $u = 1 + 4x$, $du = 4\ dx$. Subbing in we obtain

$$\begin{aligned}
\int \frac{5}{1+4x}\ dx &= \frac{5}{4}\int \frac{4\ dx}{1+4x} \\
&= \frac{5}{4}\int \frac{du}{u} \\
&= \frac{5}{4}\left(\ln|u| + C\right), \\
&= \frac{5}{4}\ln|1+4x| + C,
\end{aligned}$$

(f) Let $u = x^2 - 1$, $du = 2x\ dx$. Subbing in we obtain

$$\begin{aligned}
\int xe^{x^2-1}\ dx &= \frac{1}{2}\int 2xe^{x^2-1}\ dx, \\
&= \frac{1}{2}\int e^u\ du, \\
&= \frac{1}{2}e^u + C, \\
&= \frac{1}{2}e^{x^2-1} + C.
\end{aligned}$$

(g) Let $u = x + 1$, $du = dx$. Note also that $x - 2 = u - 3$. Subbing in we obtain

$$\begin{aligned}
\int 12(x-2)(x+1)^5\ dx &= \int 12(u-3)u^5\ du, \\
&= 12\int (u^6 - 3u^5)\ du, \\
&= 12\left(\frac{1}{7}u^7 - \frac{1}{2}u^6\right) + C, \\
&= \frac{12}{7}(x+1)^7 - 6(x+1)^6 + C.
\end{aligned}$$

(h) Let $u = x + 1$, $du = dx$. Subbing in we obtain

$$\begin{aligned}
\int \frac{x}{x+1}\ dx &= \int \frac{u-1}{u}\ du \\
&= \int \left(1 - \frac{1}{u}\right)\ du \\
&= u - \ln|u| + C \\
&= x + 1 - \ln|x+1| + C, \text{ or} \\
&= x - \ln|x+1| + C, \text{ since we can incorporate the 1 into the constant } C
\end{aligned}$$

(i) Let $u = e^{2x} + 1$, $du = 2e^{2x}\, dx$. Subbing in we obtain

$$\begin{aligned}
\int \frac{e^{2x}}{e^{2x}+1}\, dx &= \frac{1}{2}\int \frac{2e^{2x}\, dx}{e^{2x}+1}, \\
&= \frac{1}{2}\int \frac{du}{u}, \\
&= \frac{1}{2}\ln|u| + C, \\
&= \frac{1}{2}\ln|e^{2x}+1| + C.
\end{aligned}$$

(j) Let $u = e^{2x} + 1$, $du = 2e^{2x}\, dx$. Subbing in we obtain

$$\begin{aligned}
\int \frac{e^{4x}}{e^{2x}+1}\, dx &= \frac{1}{2}\int \frac{2e^{2x}e^{2x}\, dx}{e^{2x}+1}, \\
&= \frac{1}{2}\int \frac{e^{2x}}{e^{2x}+1}\, 2e^{2x}\, dx, \\
&= \frac{1}{2}\int \frac{u-1}{u}\, du, \\
&= \frac{1}{2}\int \left(1 - \frac{1}{u}\right)\, du, \\
&= \frac{1}{2}\left(u - \ln|u|\right) + C, \\
&= \frac{1}{2}\left(e^{2x} + 1 - \ln|e^{2x}+1|\right) + C, \text{ or} \\
&= \frac{1}{2}\left(e^{2x} - \ln|e^{2x}+1|\right) + C.
\end{aligned}$$

2. (a) Let $u = x^3 + 4$, $du = 3x^2\, dx$. $x = 0 \Rightarrow u = 4$ and $x = 1 \Rightarrow u = 5$. Subbing in we obtain

$$\begin{aligned}
\int_0^1 3x^2\sqrt{x^3+4}\, dx &= \int_4^5 \sqrt{u}\, du, \\
&= \frac{2}{3}u^{3/2}\Big|_4^5, \\
&= \frac{2}{3}(5^{3/2}) - \frac{2}{3}(4^{3/2}), \\
&= \frac{2\sqrt{125}}{3} - \frac{16}{3} \approx 2.12.
\end{aligned}$$

(b) Let $u = x^2 - 3$, $du = 2x\, dx$. $x = 2 \Rightarrow u = 1$ and $x = 3 \Rightarrow u = 6$. Subbing in we obtain

$$\begin{aligned}
\int_2^3 \frac{x}{x^2-3}\, dx &= \frac{1}{2}\int_2^3 \frac{2x}{x^2-3}\, dx, \\
&= \frac{1}{2}\int_1^6 \frac{du}{u}, \\
&= \frac{1}{2}\ln|x|\Big|_1^6, \\
&= \frac{1}{2}(\ln 6 - \ln 1) = \frac{\ln 6}{2} \approx 0.896.
\end{aligned}$$

(c) Let $u = x^3 + 73$, $du = 3x^2\ dx$. $x = 2 \Rightarrow u = 81$ and $x = 3 \Rightarrow u = 100$. Subbing in we obtain

$$\begin{aligned}
\int_2^3 \frac{x^2}{\sqrt{x^3+73}}\ dx &= \frac{1}{3}\int_2^3 \frac{3x^2\ dx}{\sqrt{x^3+73}}, \\
&= \frac{1}{3}\int_{81}^{100} u^{-1/2}\ du, \\
&= \frac{1}{3}(2u^{1/2})\Big|_{81}^{100}, \\
&= \frac{2}{3}(100^{1/2} - 81^{1/2}), \\
&= \frac{2}{3}(10-9) = \frac{2}{3}.
\end{aligned}$$

(d) Let $u = x^3$, $du = 3x^2\ dx$. $x = 0 \Rightarrow u = 0$ and $x = 2 \Rightarrow u = 8$. Subbing in we obtain

$$\begin{aligned}
\int_0^2 4x^2 e^{x^3}\ dx &= \frac{4}{3}\int_0^2 3x^2 e^{x^3}\ dx, \\
&= \frac{4}{3}\int_0^8 e^u\ du, \\
&= \frac{4}{3}e^u\Big|_0^8, \\
&= \frac{4}{3}(e^8 - 1) \approx 3973.28.
\end{aligned}$$

(e) Let $u = \ln x$, $du = \frac{dx}{x}$. $x = e \Rightarrow u = 1$ and $x = e^2 \Rightarrow u = 2$. Subbing in we obtain

$$\begin{aligned}
\int_e^{e^2} \frac{1}{x\ln x}\ dx &= \int_e^{e^2} \frac{1}{\ln x}\frac{dx}{x}, \\
&= \int_1^2 \frac{du}{u}, \\
&= \ln|x|\ |_1^2, \\
&= \ln 2 - \ln 1 = \ln 2 \approx 0.693
\end{aligned}$$

3. (a) When x is in the range $[0, 2]$, e^x is the "upper" function, so the area between the curves is given by

$$\int_0^2 (e^x - e^{-x})\ dx = (e^x + e^{-x})|_0^2 = (e^2 + e^{-2}) - (e^0 + e^{-0}) = e^2 + e^{-2} - 2 \approx 5.52$$

(b) We proceed as above with different limits of integration

$$\int_0^{\ln 2} (e^x - e^{-x})\ dx = (e^x + e^{-x})|_0^{\ln 2} = (e^{\ln 2} + e^{-\ln 2}) - (e^0 + e^{-0}) = (2 + \frac{1}{2}) - 2 = \frac{1}{2}.$$

(c) We need to evaluate the integral

$$\int_0^1 \left(e^x - \sqrt{x}e^{\sqrt{x^3}}\right)\ dx = \int_0^1 e^x\ dx - \int_0^1 \sqrt{x}e^{\sqrt{x^3}}\ dx$$

We will do this in two steps, evaluating each of the integrals on the right hand side separately. The first one is straight forward:

$$\begin{aligned}\int_0^1 e^x \, dx &= e^x\big|_0^1 \\ &= e^1 - e^0 = e - 1.\end{aligned}$$

To evaluate the second one, let $u = \sqrt{x^3} = x^{3/2}$, $du = \frac{3}{2}x^{1/2}$. $x = 0 \Rightarrow u = 0$ and $x = 1 \Rightarrow u = 1$. Subbing in we obtain

$$\begin{aligned}\int_0^1 \sqrt{x}e^{\sqrt{x^3}} \, dx &= \frac{2}{3}\int_0^1 \frac{3}{2}\sqrt{x}e^{\sqrt{x^3}} \, dx, \\ &= \frac{2}{3}\int_0^1 e^u \, du, \\ &= \frac{2}{3}e^u\Big|_0^1 \\ &= \frac{2}{3}(e-1)\end{aligned}$$

Therefore the area between the two curves is

$$\begin{aligned}\int_0^1 \left(e^x - \sqrt{x}e^{\sqrt{x^3}}\right) \, dx &= (e-1) - \frac{2}{3}(e-1) \\ &= \frac{1}{3}(e-1) \approx 0.573\end{aligned}$$

4. The general formula for the average value of a function $f(x)$ over an interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x) \, dx.$$

We've already calculated the value of the first four integrals in Problem 5 of the previous lab, and the last two integrals have been calculated above in Problem 2. It's now just a matter of dividing by the $(b - a)$ to get the average value.

(a) $f(x) = 12 - 3x^2$ on the interval $[1, 2]$

$$\begin{aligned}\frac{1}{2-1}\int_1^2 (12 - 3x^2) \, dx &= \left(\frac{1}{1}\right)(12x - x^3)\Big|_1^2 \\ &= 5\end{aligned}$$

(b) $f(x) = \dfrac{1}{x^3}$ on the interval $[1, 4]$

$$\begin{aligned}\frac{1}{4-1}\int_1^4 \frac{1}{x^3} \, dx &= \left(\frac{1}{3}\right)\frac{-1}{2x^2}\Big|_1^4 \\ &= \left(\frac{1}{3}\right)\frac{15}{32} = \frac{5}{32}\end{aligned}$$

(c) $f(x) = \dfrac{1}{x}$ on the interval $[1, 4]$

$$\begin{aligned} \frac{1}{4-1}\int_1^4 \frac{1}{x}\,dx &= \left(\frac{1}{3}\right)\ln|x|\Big|_1^4 \\ &= \frac{\ln 4}{3} \end{aligned}$$

(d) $f(x) = 5e^x$ on the interval $[-1, 2]$

$$\begin{aligned} \frac{1}{2-(-1)}\int_{-1}^2 5e^x\,dx &= \left(\frac{1}{3}\right)(5e^x)\Big|_1^2 \\ &= \frac{5e^2 - 5e^{-1}}{3} \end{aligned}$$

(e) $f(x) = \dfrac{x^2}{\sqrt{x^3+73}}$ on the interval $[2, 3]$ (using the substitution $u = x^3 + 73$)

$$\begin{aligned} \frac{1}{3-2}\int_2^3 \frac{x^2}{\sqrt{x^3+73}}\,dx &= \left(\frac{1}{1}\right)\frac{1}{3}(2u^{1/2})\Big|_{81}^{100} \\ &= \frac{2}{3} \end{aligned}$$

(f) $f(x) = 4x^2e^{x^3}$ on the interval $[0, 2]$ (using the substitution $u = x^3$)

$$\begin{aligned} \frac{1}{2-0}\int_0^2 4x^2e^{x^3}\,dx &= \left(\frac{1}{2}\right)\frac{4}{3}e^u\Big|_0^8, \\ &= \frac{2}{3}(e^8 - 1) \approx 1986.64. \end{aligned}$$