MTH131 Applied Calculus – Spring 2016
Lab 3 – Derivatives
SOLUTIONS

1. (a) $f'(x) = 4x^3$
   (b) $g'(x) = \frac{3}{2}x^2$
   (c) $g'(w) = 2w^{-2/3}$
   (d) $h'(x) = -\dfrac{6}{x^3}$
   (e) $f'(x) = 8x - 3$
   (f) $f'(r) = 2\pi r$
   (g) $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{x^2}$

2. In all there problems, you can stop simplifying when you get to lines marked with a (*).
   (a) $f(x) = x^3(x^2+1)$: Using the product rule

$$\begin{aligned} f'(x) &= 3x^2(x^2+1) + x^3(2x) \\ &= 3x^2(x^2+1) + 2x^4 \quad (*) \\ &= 3x^4 + 3x^2 + 2x^4 \\ &= 5x^4 + 3x^2 \end{aligned}$$

   (b) $g(x) = (x^3-1)(x^3+1)$: Using the product rule

$$\begin{aligned} g'(x) &= 3x^2(x^3+1) + (x^3-1)(3x^2) \quad (*) \\ &= 3x^5 + 3x^2 + 3x^5 - 3x^2 \\ &= 6x^5 \end{aligned}$$

   (c) $g(t) = (2t-1)(1-t^2)$: Using the product rule

$$\begin{aligned} g'(t) &= 2(1-t^2) + (2t-1)(-2t) \\ &= 2(1-t^2) - (2t-1)(2t) \quad (*) \\ &= 2 - 2t^2 - 4t^2 + 2t \\ &= -6t^2 + 2t + 2 \end{aligned}$$

   (d) $f(x) = x^3(x^2-4x+3)$: Using the product rule

$$\begin{aligned} f'(x) &= 3x^2(x^2-4x+3) + x^3(2x-4) \quad (*) \\ &= 3x^4 - 12x^3 + 9x^2 + 2x^4 - 4x^3 \\ &= 5t^4 - 16x^3 + 9x^2 \end{aligned}$$

   (e) $f(x) = (x+6\sqrt{x})(x-2\sqrt{x}+1) = (x+6x^{1/2})(x-2x^{1/2}+1)$: Using the product rule

$$\begin{aligned} f'(x) &= (1+6(1/2)x^{-1/2})(x-2x^{1/2}+1) + (x+6x^{1/2})(1-2(1/2)x^{-1/2}) \\ &= (1+3x^{-1/2})(x-2x^{1/2}+1) + (x+6x^{1/2})(1-x^{-1/2}) \quad (*) \\ &= (x+x^{1/2}-5+3x^{-1/2}) + (x+5x^{1/2}-6) \\ &= 2x+6x^{1/2}-11+3x^{-1/2} \end{aligned}$$

(f) $h(x) = \dfrac{x^5 - 1}{x^2}$: Using the quotient rule:

$$\begin{aligned} h'(x) &= \frac{x^2(5x^4) - (x^5-1)(2x)}{(x^2)^2} \\ &= \frac{5x^6 - (x^5-1)(2x)}{x^4} \quad (*) \\ &= \frac{5x^6 - 2x^6 + 2x}{x^4} \\ &= \frac{3x^6 + 2x}{x^4} \end{aligned}$$

(g) $g(t) = \dfrac{t^2 + 1}{t^2 - 1}$: Using the quotient rule:

$$\begin{aligned} g'(t) &= \frac{(t^2-1)(2t) - (t^2+1)(2t)}{(t^2-1)^2} \quad (*) \\ &= \frac{2t^3 - 2t - 2t^3 - 2t}{(t^2-1)^2} \\ &= -\frac{4t}{(t^2-1)^2} \end{aligned}$$

(h) $f(x) = \dfrac{x^5 + x^3 + x}{x^3 + x}$: Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x^3+x)(5x^4+3x^2+1) - (x^5+x^3+x)(3x^2+1)}{(x^3+x)^2} \quad (*) \\ &= \frac{(5x^7+8x^5+4x^3+x) - (3x^7+4x^5+4x^3+x)}{(x^3+x)^2} \\ &= \frac{2x^7+4x^5}{(x^3+x)^2} \end{aligned}$$

(i) $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$: Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{(x^{1/2}-1)(\frac{1}{2}x^{-1/2}) - (x^{1/2}+1)(\frac{1}{2}x^{-1/2})}{(x^{1/2}-1)^2} \quad (*) \\ &= \frac{(1/2 - \frac{1}{2}x^{-1/2}) - (1/2 + \frac{1}{2}x^{-1/2})}{(\sqrt{x}-1)^2} \\ &= -\frac{x^{-1/2}}{(\sqrt{x}-1)^2} \\ &= -\frac{1}{\sqrt{x}(\sqrt{x}-1)^2} \end{aligned}$$

(j) $f(x) = \dfrac{(x^5+1)(x^3+2)}{x+1}$: For this problem, you must use both the product and quotient rules. You can do them in either order. The answer below uses the quotient rule first, then the product rule.

$$\begin{aligned}
f'(x) &= \frac{(x+1)\frac{d}{dx}[(x^5+1)(x^3+2)] - (x^5+1)(x^3+2)\frac{d}{dx}[x+1]}{(x+1)^2} \\
&= \frac{(x+1)[(5x^4)(x^3+2)+(x^5+1)(3x^2)] - (x^5+1)(x^3+2)}{(x+1)^2} \quad (*) \\
&= \frac{(x+1)[(5x^7+10x^4)+(3x^7+3x^2)] - (x^8+2x^5+x^3+2)}{(x+1)^2} \\
&= \frac{(x+1)(8x^7+10x^4+3x^2) - (x^8+2x^5+x^3+2)}{(x+1)^2} \\
&= \frac{(8x^8+8x^7+10x^5+10x^4+3x^3+3x^2) - (x^8+2x^5+x^3+2)}{(x+1)^2} \\
&= \frac{7x^8+8x^7+8x^5+10x^4+2x^3+3x^2-2}{(x+1)^2}
\end{aligned}$$

3. The derivative of $f(x) = \dfrac{x+1}{x-1}$ is

$$\begin{aligned}
f'(x) &= \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} \\
&= -\frac{2}{(x-1)^2}.
\end{aligned}$$

We want to know where this function is $= -2$:

$$\begin{aligned}
-\frac{2}{(x-1)^2} &= -2 && \text{divide by } -2 \\
\frac{1}{(x-1)^2} &= 1 && \text{invert} \\
(x-1)^2 &= 1 && \text{expand} \\
x^2-2x+1 &= 1 \\
x^2-2x &= 0 && \text{factor, and solve for } x \\
x(x-2) &= 0 \\
x &= 0, 2
\end{aligned}$$

At $x = 0$ the function will have the value $f(0) = (0+1)/(0-1) = -1$. Given that the slope $= -2$ at this point, we use the point-slope formula for a line to get the equation of the tangent line:

$$\begin{aligned}
(y-(-1)) &= -2(x-0), \\
y+1 &= -2x, \\
y &= -2x-1.
\end{aligned}$$

At $x = 2$ the function will have the value $f(2) = (2 + 1)/(2 - 1) = 3$. Given that the slope $= -2$ at this point, we use the point-slope formula for a line to get the equation of the tangent line:

$$\begin{aligned} (y - 3) &= -2(x - 2), \\ y - 3 &= -2x + 4, \\ y &= -2x + 7. \end{aligned}$$

4. (a) The rate of change of temperature is given by the derivative $f'(t) = 8t^{1/3}$. We need to know when this is equal to 16 degress per minute:

$$\begin{aligned} 8t^{1/3} &= 16, \\ t^{1/3} &= 2, \\ t &= 8. \end{aligned}$$

So after 8 minutes, the temperature is changing at 16 degress per minute.

(b) It takes 8 minutes to reach the point when the temperature is changing at 16 degrees/minute. At that point, the actual temperature is $f(8) = 24 + 6(8)^{4/3} = 120$. If the metal is now cooled at 10 degrees per minute, it takes 12 more minutes to get to 0 degress Celcius. Therefore, the entire process takes $8+12 = 20$ minutes.

5. (a) $f(x) = x^3(x^2 + 1)$: Simplifying $f(x)$ first

$$\begin{aligned} f(x) &= x^5 + x^3, \\ f'(x) &= 5x^4 + 3x^2 \end{aligned}$$

(b) $g(x) = (x^3 - 1)(x^3 + 1)$: Simplifying $g(x)$ first

$$\begin{aligned} g(x) &= x^6 - 1 \quad \text{using rule } (a + b)(a - b) = a^2 - b^2, \\ g'(x) &= 6x^5 \end{aligned}$$

(c) $g(t) = (2t - 1)(1 - t^2)$: Simplifying $g(t)$ first

$$\begin{aligned} g(t) &= -2t^3 + t^2 + 2t - 1, \\ g'(t) &= -6t^2 + 2t + 2 \end{aligned}$$

(d) $f(x) = x^3(x^2 - 4x + 3)$: Simplifying $f(x)$ first

$$\begin{aligned} f(x) &= x^5 - 4x^4 + 3x^3 \\ f'(x) &= 5x^4 - 16x^3 + 9x^2 \end{aligned}$$