MTH131 Applied Calculus – Spring 2016
Lab 2 – Instantaneous Rates of Change and the Derivative
SOLUTION

1. (a) $\lim_{x\to 0} \frac{x+2}{x-2} = \frac{0+2}{0-2} = \frac{2}{-2} = -1$

   $\lim_{x\to 2} \frac{x+2}{x-2} = \frac{2+2}{2-2} = \frac{4}{0} \Rightarrow$ DNE

   (b) $\lim_{x\to 0} \frac{x^2-4}{x-2} = \frac{0^2-4}{0-2} = \frac{-4}{-2} = 2$

   $\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{2^2-4}{2-2} = \frac{0}{0} \Rightarrow$ we must do more work.

   We use that fact that $x^2-4=(x+2)(x-2)$ to simplify the expression:

   $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$

   (c) $\lim_{x\to 0} \frac{x^2-4}{(x-2)^2} = \frac{0^2-4}{(0-2)^2} = \frac{-4}{(-2)^2} = -1$

   $\lim_{x\to 2} \frac{x^2-4}{(x-2)^2} = \frac{2^2-4}{(2-2)^2} = \frac{0}{0^2} \Rightarrow$ we must do more work.

   Using $x^2-4=(x+2)(x-2)$ we have:

   $\lim_{x\to 2} \frac{x^2-4}{(x-2)^2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)^2} = \lim_{x\to 2} \frac{x+2}{x-2} = \frac{2+2}{2-2} = \frac{4}{0} \Rightarrow$ DNE

2. (a) We use the definition of average rate of change:

$$\begin{aligned}\frac{\Delta f}{\Delta x} &= \frac{f(3)-f(0)}{3-0}\\ &= \frac{2^3-2^0}{3-0}\\ &= \frac{8-1}{3-0}\\ &= \frac{7}{3}\end{aligned}$$

   (b) The chart below shows the approximation of instantaneous rate on the interval $[1, 1+h]$ for smaller and smaller h values: :

h	$\frac{f(1+h)-f(1)}{h} = \frac{2^{1+h}-2^1}{h}$
0.1	1.435
0.01	1.391
0.001	1.387
0.0001	1.386
0.00001	1.386

   It turns out the exact answer is $2\ln 2 = 1.386294$, which we will prove later in the course. (If you don't know what $\ln 2$ means, we'll be explaining that later in the course as well).

3. (a) We use the definition of average rate of change:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

In both cases $x_1 = 9$ and $x_2 = 10$ (i.e., 9AM and 10AM).

i. When $f(x) = 5x + 25$ we have

$$\begin{aligned}\frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{5(10) + 25 - [5(9) + 25]}{10 - 9} \\ &= \frac{75 - 70}{1} \\ &= 5.\end{aligned}$$

ii. When $y = f(x) = 2x^2 - 35x + 217$ we have

$$\begin{aligned}\frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{2(10)^2 - 35(10) + 217 - [2(9)^2 - 35(9) + 217]}{10 - 9} \\ &= \frac{67 - 64}{1} \\ &= 3.\end{aligned}$$

(b) We use the definition of instantaneous rate of change

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

In both cases we want the instantaneous rate of change when $c = 9$ so the formula becomes:

$$\lim_{h \to 0} \frac{f(9 + h) - f(9)}{h}$$

i. When $y = f(x) = 5x + 25$ we have

$$\begin{aligned}\lim_{h \to 0} \frac{f(9 + h) - f(9)}{h} &= \lim_{h \to 0} \frac{5(9 + h) + 25 - [5(9) + 25]}{h} \\ &= \lim_{h \to 0} \frac{45 + 5h + 25 - [45 + 25]}{h} \\ &= \lim_{h \to 0} \frac{5h}{h} \\ &= \lim_{h \to 0} 5 \\ &= 5\end{aligned}$$

ii. When $y = f(x) = 2x^2 - 35x + 217$ we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(9+h)-f(9)}{h} &= \lim_{h\to 0} \frac{2(9+h)^2 - 35(9+h) + 217 - [2(9)^2 - 35(9) + 217]}{h} \\
&= \lim_{h\to 0} \frac{2(81 + 18h + h^2) - 35(9) - 35h + 217 - [2(81) - 35(9) + 217]}{h} \\
&= \lim_{h\to 0} \frac{2(18h + h^2) - 35h}{h} \\
&= \lim_{h\to 0} \frac{36h + 2h^2 - 35h}{h} \\
&= \lim_{h\to 0} \frac{h + 2h^2}{h} \\
&= \lim_{h\to 0} 1 + 2h \\
&= 1
\end{aligned}$$

(c) We use the definition of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

i. When $y = f(x) = 5x + 25$ we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{5(x+h) + 25 - [5x + 25]}{h} \\
&= \lim_{h\to 0} \frac{5x + 5h + 25 - [5x + 25]}{h} \\
&= \lim_{h\to 0} \frac{5h}{h} \\
&= \lim_{h\to 0} 5 \\
&= 5
\end{aligned}$$

Using this formula we have $f'(9) = 5$, which is what we got in part (b). At 10AM the rate of change will be $f'(10) = 5$ (note that the derivative in this case is just a constant function).

ii. When $y = f(x) = 2x^2 - 35x + 217$ we have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{2(x+h)^2 - 35(x+h) + 217 - [2x^2 - 35x + 217]}{h} \\
&= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) - 35x - 35h + 217 - [2x^2 - 35x + 217]}{h} \\
&= \lim_{h\to 0} \frac{2(2xh + h^2) - 35h}{h} \\
&= \lim_{h\to 0} \frac{4xh + 2h^2 - 35h}{h} \\
&= \lim_{h\to 0} 4x + 2h - 35 \\
&= 4x - 35
\end{aligned}$$

Using this formula we have $f'(9) = 4(9) - 35 = 1$, which is what we got in part (b). At 10AM the rate of change will be $f'(10) = 4(10) - 35 = 5$.